TD Interest rates

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1 The Ornstein-Uhlenbeck process

Let

$$dx(t) = k(\theta - x(t))\, dt + \sigma\, dW_t \quad (1)$$
$$x(0) = r_0 \quad (2)$$

where W_t is a SBM. The standard deviation parameter, σ, determines the volatility of the interest rate. The typical parameters θ, k and σ, together with the initial condition r_0.

1.1 EWMA* of Brownian motion

*: **E**xponentially **W**eighted **M**oving **A**verage

Let

$$I(t,T) := \int_t^T e^{-k(T-s)} dW_s$$

Show that:

$$\mathbb{E}(I(t,T) \mid \mathcal{F}) = 0$$
$$Var(I(t,T) \mid \mathcal{F}) = A(2k, T-t)$$

with $A(k,u) := \frac{1}{k}\left(1 - e^{-ku}\right)$

Study the asymptotic behavior of $V(t,T)$ when $T \to t$ and $T \to +\infty$

1.2 Soving the SDE

Define

$$r(T) = \theta + (r_t - \theta)\, e^{-k(T-t)} + \sigma I(t,T)$$

Now:

1. Using Ito's lemma, check that $r(t)$ is a solution of equation (1)

2. Thanks to change of variables show that equation (1) can be integrated with $r(t)$ as a solution.

Then show that:

$$\mathbb{E}\left(r_T|r_t\right) = \theta + \left(r_t - \theta\right) e^{-k(T-t)}$$
$$Var\left(r_T|r_t\right) = V\left(t,T\right) = \sigma^2 A(2k, T-t)$$

$$\begin{aligned}\gamma\left(t,s,u\right) &= cov\left(r_s, r_u|r_t\right)\\ &= \mathbb{E}\left(I\left(t,s\right)I\left(t,u\right)|r\left(t\right)\right)\\ &= \frac{\sigma^2}{2k}e^{-k(s+u)}\left(e^{2ku\wedge s} - e^{2kt}\right)\end{aligned}$$

2 The integrated Short Term rate

Define:

$$\xi(t,T) = \int_t^T r(s)ds$$

2.1 Expectation

Show that:

$$\mathbb{E}\left(\xi\left(t,T\right)|r_t\right) = \theta\left(T-t\right) + \left(r\left(t\right) - \theta\right)A(k, T-t)$$

2.2 Variance

Show that:

$$Var\left(\xi\left(t,T\right)|r_t\right) = \int_t^T \int_t^T \gamma\left(t,s,u\right)dsdu$$

Then show that

$$Var\left(\xi\left(t,T\right)|r_t\right) = \frac{\sigma^2}{k^2}\int_t^T \left[1 - 2e^{-k(u-t)} + e^{-2k(u-t)}\right]du$$

Eventually show that:

$$V\left(t,T\right) = \frac{\sigma^2}{2k^3}\left[2k\left(T-t\right) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)}\right]$$

3 Zero-coupon rate

3.1 Formula

The zero-coupon price can be calculated as:

$$P(t,T) = \mathbb{E}\left(e^{-\xi(t,T)} \mid \mathcal{F}\right)$$

The zero-coupon rate is defined as

$$R\left(t,T\right) \equiv \frac{1}{T-t} ln\left(P\left(t,T\right)\right)$$

Show that

$$R\left(t,T\right) = \bar{A}(k,T-t)r(t) + B(t,T) \tag{3}$$

with

$$\begin{aligned}
\bar{A}(k,T-t) &= \frac{1}{T-t} A(k,T-t) \\
(T-t)B\left(t,T\right) &= \left(\theta - \frac{\sigma^2}{2k^2}\right)\left(A(k,T-t) - (T-t)\right) - \frac{\sigma^2}{4k} A^2(k,T-t)
\end{aligned}$$

3.2 Covariance structure

Show that

$$d\left\langle r(t,T_1), r(t,T_2))\right\rangle = \sigma^2 \bar{A}(k,T_1-t)\bar{A}(k,T_2-t)dt$$

Deduce that the instantaneous correlation between $dr(t,T_1)$ and $dr(t,T_2)$ is equal to:

$$\begin{aligned}
\rho(r(t,T_1), r(t,T_2)) &:= \frac{d\left\langle r(t,T_1), r(t,T_2)\right\rangle}{\sqrt{d\left\langle r(t,T_1)\right\rangle d\left\langle r(t,T_2)\right\rangle}} \\
&= 1
\end{aligned}$$

Comment on the covariance structure of the yield curve in a one factor Vasicek Model