UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Financial Products and Introduction to Pricing

MMMEF & IRFA
Prof. : Nicolas Gaussel
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Contents

1 General concepts of finance and accounting

1.1 Quick introduction about financial markets

Financial markets are a variety of places where those who need money meet those who save money. But the "match" cannot be straightforward. There are several families of market participants.

1.1.1 Those who need money

Mostly, states, corporations and households:

- World market cap: ~ 100 Tn
- World corporate debt: ~ 90 Tn
- World sovereign debt: ~ 90 Tn
- Households: mostly "contractual" loans. Mortgage Backed Securities market in the US (~ 12 Tn)

Roughly 3×10^{14} USD.

1.1.2 Those who save money

- Asset Managers: ~ 120 Tn (half of which equity funds)
- Insurance companies: ~ 40 Tn
- Sovereign funds: ~ 10 Tn
- Endowments and foundations: ~ 2 Tn

1.1.3 Intermediaries

Banks, asset managers, etc.

The added value of financial markets

- Liquidity
- Transparency (disclosure rules)
- Normalisation
- Risk transfer

Notion of "security" From bespoke contracts to normalized transferable securities.

1.2 Double entry accounting system

In the double-entry accounting system, at least two accounting entries are required to record each financial transaction...

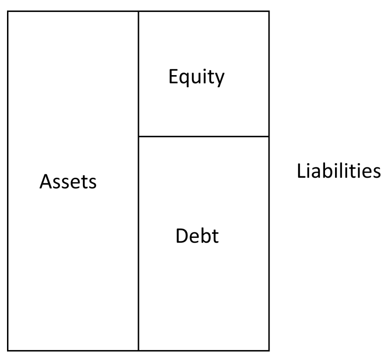


Figure 1: Asset and Liability representation

Proposition 1. *At any moment in time, for any balance sheet:*

$$Assets = Liabilities.$$

1.3 Joint dynamics of Assets, Equity and Debts

Let A_t, E_t, D_t the respective level of assets, equity and debt at time t.

Definition 1 (Leverage)**.** *The ratio between assets and equity is called* ***leverage:*** $l_t \equiv \frac{A_t}{E_t}$

Between t and $t + \Delta t$ the equality between assets and liabilities imposes that

$$\Delta A_t = \Delta E_t + \Delta D_t$$

The relative variation of wealth E can be written as

$$\begin{aligned}\frac{\Delta E_t}{E_t} &= \frac{\Delta A_t}{E_t} - \frac{\Delta D_t}{E_t} \\ &= \frac{A_t}{E_t}\frac{\Delta A_t}{A_t} - \frac{D_t}{E_t}\frac{\Delta D_t}{D_t} \\ &= l_t \frac{\Delta A_t}{A_t} - (l_t - 1)\frac{\Delta D_t}{D_t}\end{aligned}$$

or equivalently

$$\frac{\Delta E_t}{E_t} - \frac{\Delta D_t}{D_t} = l_t \left(\frac{\Delta A_t}{A_t} - \frac{\Delta D_t}{D_t} \right)$$

The previous accounting equation shows that the excess return of Equity over debt is proportionnal to the leverage times the excess of returns of assets over debt.

If one models the premia of equity and assets as stationary random variables $\tilde{\pi}_E \equiv \frac{\Delta E_t}{E_t} - \frac{\Delta D_t}{D_t}$ and $\tilde{\pi}_A \equiv \frac{\Delta A_t}{A_t} - \frac{\Delta D_t}{D_t}$, then the following linear relationship between returns holds

$$\tilde{\pi}_E = l \times \tilde{\pi}_A$$

As a result

$$\pi_E = E(\tilde{\pi}_E) = l\pi_A$$
$$\sigma_E = stdev(\tilde{\pi}_E) = l\sigma_A$$

Definition 2. *The extra return of an asset above the cost of financing is called the* ***risk premium*** *of that assets. The (forward looking) average of the risk premium the "****Expected Risk Premium****" or "****Expected Excess Return****". The standard deviation of the return of an asset is called its* ***volatility****.*

The above equations entail the following consequences:

1. An asset which return is lower than the return of debt has a negative risk premia
2. The expected risk premia as welle as the volatility of an asset are linearly amplified by the leverage

Remark 1. *In finance, stocks are expressed in currencies. Relative variations such as π are expressed in inverse of time (per annum, per day, etc.). Their dimension is thus T^{-1}. Hence, one has to be careful about the time units which has been chosen. In many instances, the "hidden" time unit is the year. When interest rates are said to be equal to 5% it implictly means "5% per year". With that backdrop, a day is equal to 1/256 or 1/365 or something else depending on the corresponding convention.*

1.4 Limited liabilities

To foster business development, the concept of limite liabilities has emerged. It states that **the value of Equity shall always remain positive**. In the case where either the company can no longer pay its bills or the value of assets falls below the value of debts, then the company is said to be in bankrupcy. In that case where E should negative, its value is floored at 0. At that moment it is the value of debt which is impacted and the equity holder might not expect to lose more than what she had invested.

We will come back later on the impact on fhis floor on equity and debt valuation but we can already make some remarks regarding leverage and bankrupcy risk. What variation of the asset can lead to bankrupcy ? If we neglect the impact of debt:

$$\begin{aligned}&\pi_E \leq -100\% \\ \Longleftrightarrow &l\pi_A \leq -1 \\ \Longleftrightarrow &\pi_A \leq -\frac{1}{l}\end{aligned}$$

Examples:....

2 FIC Forward contracts

2.1 Linear vs continuous rates

In this section, the interest rate is assumed to be constant equal to r. **Interest rates are almost always expressed per annum** in order to be compared properly. A flow of money is defined by a currency and a date of payment. While it seems obvious that $1\ USD \neq 1\ EUR$ it is generally less obvious than $1\ USD\ today \neq 1\ USD\ tomorrow$. Think about the promise of principal protected products: is it really an appealing deal ?

Foreign Exchange Rates are the adjustment factor between the value of two unity of currencies. **Interest rates** are the adjustment factor between the value of two unity of the same currency paid at different dates. The equalisation of date 1 value with date two value is called **Compounding** when we move forward in time and **discounting** when we move backward in time.

Due to the interest paid on deposits, the compounding factor is generally greater than 1 while the discounting factor is generally lower than 1.

Let us now try to give a precise meaning to the above. If $r = 5\%$, what is the value of 1 paid in $t = 2Y$?

If compounding is linear then the compounding factor is equal to

$$C = 1 + rt$$

. But it could be agreed that compounding happens twice during the period and one would have

$$C = (1 + \frac{rt}{2}) \times (1 + \frac{rt}{2})$$

which would become

$$C(n) = (1 + \frac{rt}{n})^n$$

if the time interval were to be divided n times.

This shows that the relationship between the compounding factor and the level of the interest rates only makes sense if a compounding frequency has been agreed upon. The choice of the compounding frequency is difficult because it represents a lower limit under which discounting can be made. If one agree on a monthly compounding how can weekly compounding can be made ?

An answer to that is to look at the limit of continuous compounding i.e. when $n \to +\infty$.

$$C(t, n) = (1 + \frac{rt}{n})^n$$
$$C(t) = \lim_{n\to\infty} C(t, n) = e^{rt}$$

Discounting being the inverse operation of compounding, the value, seen from 0, of a flow of 1 at date t is equal to $B_t = C_t^{-1}$. The compounding and discounting factors can be interpreted as **the exchange rate between two dates**.

In the actuarial literature one generally find $B_t = (1 + r_a)^{-t}$ where r_a stands for the actuarial rate. While the exponential is more clear and allow more tractability, both definition are equivalent indeed with $r_a = e^r - 1$.

2.2 Compounding/discouting

Definition 3 (Zero Coupon Bond)**.** *A security that would pay 1 some date is called a zero-coupon Bond. The value at date t of the zero-coupon paying 1 at date T is denoted $P(t, T)$*

.

Definition 4 (Net Present Value)**.** *Consider a stream of cash-flows X_τ where $\tau \equiv t_1, ..., t_n$. The Net Present Value of X_τ is defined by:*

$$NPV(t) = \sum_i P(t, t_i) \times X_i$$

This NPV can be "transported" at any other date t_k:

$$NPV(t_k) = NPV(t) \times \frac{1}{P(t, t_k)}$$

Definition 5 (Duration).

$$\frac{\partial P(t,r)}{\partial r} = -tP(t,r)$$
$$\Leftrightarrow \frac{\partial P(t,r)}{P(t,r)\partial r} = -t$$

An instantaneous variation of the rate has consequences during the entire life of the bond.

Example 1 (Bullet bond). *Compute the value of a bond which pays an interest rate r∗ every year and redeem all the capital at the end, when interest rates are equal to r*

$$B(r^*, r) = r^* \sum_1^n \frac{1}{(1+r)^i} + \frac{1}{(1+r)^n} \quad (1)$$
$$= \frac{r^*}{r}(1 - \frac{1}{(1+r)^n}) + \frac{1}{(1+r)^n} \quad (2)$$

You can note that :

- $r^* > r \Rightarrow \; B(r^*, r) > 1$ *("above par")*
- $r^* < r \Rightarrow \; B(r^*, r) < 1$ *("below par")*
- $r^* = r \Rightarrow \; B(r^*, r) = 1$ *("at par")*

2.3 Forward Rate Agreements

Let $R(t,T) := -\frac{ln(P(t,T))}{T-t}$ be the continuously compounded zero-coupon rate and $L(t,S)$ the linearly compounded rate defined by $P(t,S) = (1 + (S-T)\,L(t,S))^{-1}$.

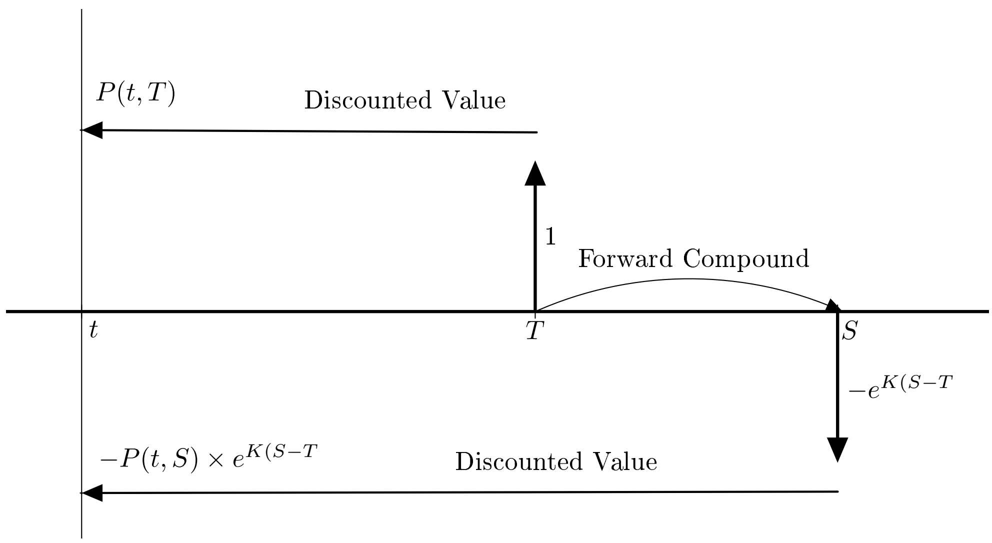
Definition 6. *The zero-coupon curve (sometimes also referred to as "yield curve") at time t is the graph of the function*

$$T \to \begin{cases} L(t,T) & t < T \leq t+1 (\textit{Short Term}) \\ R(t,T) & T > t+1 (\textit{Long Term}) \end{cases}$$

.

Definition 7. *A Forward Rate Agreement (FRA) is a contract involving three time instants: The current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest-rate payment for the period between T and S. At the maturity S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $R(T,S)$ resetting in T and with maturity S.*

The flow generated by this deal are summarized in the following figure:



Since this contract does not entail any initial flow then, necessarily:

$$P(t,T) = P(t,S)\,e^{K(S-T)}$$

or equivalently

$$F(t,T,S) \equiv K = \frac{1}{S-T} ln\left(\frac{P(t,T)}{P(t,S)}\right)$$

$F(t,T,S)$ is the continuously compounded forward rate. $F(t,T,S)$ is such that

$$R(t,S) = \frac{R(t,T)(T-t) + F(t,T,S)(S-T)}{S-t}$$

One can also define the instantaneous forward rate :

$$f(t,T) := \lim_{S \to T^+} F(t,T,S) = -\frac{\partial lnP(t,T)}{\partial T}$$

. Using the previous formula, the zero-coupon bond can be computed/interpreted as the accumulation of forward rates:

$$P(t,T) = exp\left(-\int_t^T f(t,u)\, du\right)$$

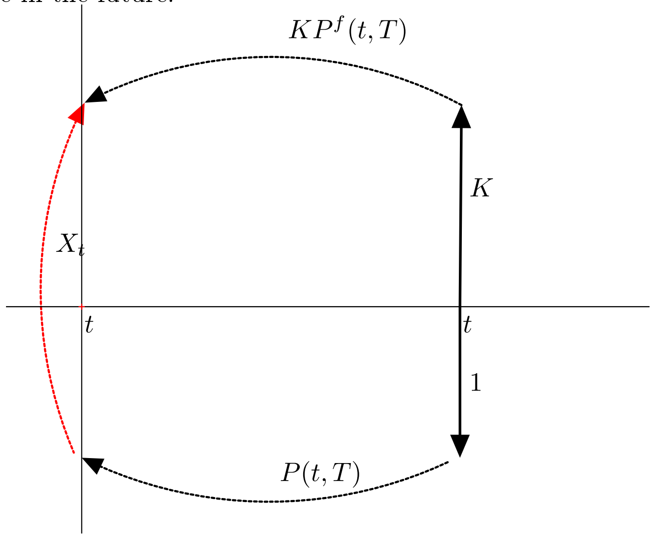
Example 2. *Flat rate curve. The forward as weighted average between two rates.*
Exercises....

2.4 Forward Exchange Rate

We keep the same notation and denote $.^f$ those quantities which are foreign quantities. For instance $P^f(t,T)$ is equal to to the foreign zero-coupon bond. Let X_t the exchange rate between two currencies meaning that 1 unit of local currency at date t is worth X_t units of foreign currency.

If X increases, the local currency appreciates, if it decreases, it depreciates.

A forward exchange rate agreement is an agreement where by one can lock today the value of an exchange rate in the future.



By the law of one price :

$$\begin{aligned} K(t,T) &= X_t \frac{P(t,T)}{P^f(t,T)} \\ &= X_t e^{\left(R^f(t,T) - R(t,T)\right)(T-t)} \end{aligned}$$

So currency shall appreciate if foreign rate is higher than local rate.

Example 3. *Multi currency business plan: what should the forward rate be ?*

3 Equity forward contract

3.1 One date Cash and Carry arbitrage

Exchange a known amount vs the T value of a stock. Idea is to buy the stock at date zero and unwind. P&L at date T is

$$(S_T - K) - \left(S_T - \frac{S_t}{P(t,T)}\right) = \frac{S_t}{P(t,T)} - K$$

This P&L is non random and can be acquired at zero price. If it is different from zero, that would be an arbitrage hence:

$$\begin{aligned} K = F\left(t,T\right) &= \frac{S_t}{P\left(t,T\right)} \\ &= S_t e^{\tau R(t,T)} \\ &\sim S_t e^{r(T-t)} \end{aligned}$$

Definition 8. *The value $F\left(S,T\right) = \frac{S_t}{P(t,T)}$ is called the forward value of the asset. the difference between the forward value and the spot value is called the* ***basis.***

The relative variation of forward price is equal to the relative variations of spot prices minus the relative variation of ZC bond:

$$\begin{aligned} dF_t &= d\left(\frac{S_t}{P\left(t,T\right)}\right) \\ &= \ldots \\ &= F_t\left(\frac{dS_t}{S_t} - \frac{dP_t}{P_t}\right) \\ &\sim F_t\left(\mu - r\right)dt \end{aligned}$$

3.2 Intermediate dividends

Consider a situation where intermediate dividends are paid at at dates $(T_1, ..., T_n)$ proportional, proportionally to the forward value at those dates: $\left(\frac{S_t}{P(t,T_1)}d_1, ..., \frac{S_t}{P(t,T_n)}d_n\right)$. In that situation, the need for financing has decreased thanks to the amount of dividend that have been paid in the course of carrying the asset:

$$\begin{aligned} F\left(t,T\right) &= \frac{S_t}{P\left(t,T\right)} - \sum_{i=1}^{i=n} \frac{S_t d_i}{P\left(t,T_i\right)} \frac{P\left(t,T_i\right)}{P\left(t,T\right)} \\ &= \frac{S_t}{P\left(t,T\right)}\left(1 - \sum d_i\right) \end{aligned}$$

4 Futures contracts

4.1 Introduction

For a detailed description of futures contract markets functioning see chapter 2 of [2]. This section follows ([1]).

While Forward contract exists in OTC markets, mostly in FX, other contracts called "Futures" contracts are dominantly traded. The trading of Futures contracts involves a mechanism called "margining". While they are essentially very similar, the margining mechanism changes the nature of their dynamics. As we will see, the difference between Forward any Futures prices is very much like the distinction between buying a zero-coupon bond and compounding short-term placement in the bond market. Let S_t be the underlying price at date t.

4.2 Characterisation of the Future price

Let $t_0 = t, t_1, .., t_n = T$ a series of dates, typically days and Let us adapt the preceding strategy in the case of Future contracts. Denote H_t the future price. The margining mechanism ensures that when the future price changes, between t_i and t_{i+1}, the party long the futures contract receives at date t_{i+1} $H(t_{i+1}) - H(t_i)$. At maturity, $H(T) = S(T)$

The Future differs from the Forward price because each variations t_{i+1} $H(t_{i+1}) - H(t_i)$ will have to be reinvested up to maturity at an unknown rate. So the future price cannot be obtained straightforwardly as functions of observable quantities as it is the case for Forward prices.

In order to better understand the nature of Future prices, we will describe a strategy which initial value is the future price and which final value can be well described. To do so, we identify such strategy for Forward contracts and try to extend it to Futures contracts.

4.2.1 Characterization of Forward Prices

At date t, the forward price of maturity T can be interpreted as the spot price of $1/P(t,T)$ units of stocks. Put differently:

Proposition 2. *The forward price is the spot price of a payoff paying* $\frac{S(T)}{P(t,T)}$ *at date* T.

Another way to think about that result is to envisage the following strategy:

1. go long $\frac{1}{P(t,T)}$ forward contracts at date t and sell them at maturity.
2. Invest the equivalent of the forward price $F(t,T)$ into a ZC bond

The initial cost of this strategy is equal to $F(t,T)$ and its value at date T is equal to

$$\begin{aligned}\frac{F(t,T)}{P(t,T)} + \frac{1}{P(t,T)}(F(T,T) - F(t,T)) &= \frac{F(T,T)}{P(t,T)} \\ &= \frac{S(T)}{P(t,T)}\end{aligned}$$

And the previous proposition can be obtained by the law of one price.

4.2.2 Extension to Future prices

In the case of future prices, the idea is to accumulate the previous strategy from t_i to t_{i+1} up to T. More precisely:

1. At t_0 go long $\frac{1}{P(0,1)}$ future contracts and invest $H(0)$ in the t_1 riskless account.
2. At t_1 unwind the strategy. The corresponding wealth will be equal to

$$\frac{1}{P(0,1)}(H(1) - H(0)) + \frac{H(0)}{P(0,1)} = \frac{H(1)}{P(0,1)}$$

At t_1, reiterate the same strategy but taking with $\frac{H(1)}{P(0,1)}$ (and not $H(0)$) as initial wealth:

1. At t_1 go long $\frac{1}{P(0,1)P(1,2)}$ future contracts and invest $H(1)/P(0,1)$ in the t_2 riskless account.
2. At date t_2 unwind the strategy. The corresponding wealth will be equal to

$$\frac{1}{P(0,1)P(1,2)}(H(2) - H(1)) + \frac{H(1)}{P(0,1)P(1,2)} = \frac{H(2)}{P(0,1)P(1,2)}$$

Let now introduce the accumulated bank account between t_i and t_j: $B(t_i,t_j) := \prod_{k=i}^{j-1} P(k,k+1)$. At each t_i the strategy consists in:

1. at date t_i go long $\frac{1}{B(t_0,t_i)}$ future contracts and invest $\frac{H(i)}{B(t_0,t_i)}$ in the t_{i+1} riskless account.
2. At date t_{i+1} unwind the strategy. The corresponding wealth will be equal to

$$\frac{1}{B(t_0,t_i)}(H(i+1) - H(i)) + \frac{H(i)}{B(t_0,t_i) \times P(i,i+1)} = \frac{H(i+1)}{B(t_0,t_{i+1})}$$

Iterating until date $t_n = T$, where $H(T) = S(T)$ it is clear that the final value of this strategy is equal to

$$\frac{S(T)}{B(t_0,T)}$$

Hence by the law of one price:

Proposition 3. *The future price is the spot price of a payoff paying* $\frac{S(T)}{B(t,T)}$ *at date* T.

The future and forward price look very much alike. Both can be understood as the spot price amplified by the cost of carrying the asset. However, for the forward price, the cost of carrying is known at the very beginning while it is random for future prices.

Proposition 4. *When interests rates are deterministic,* $B(t,T) = P(t,T)$ *and future and foward price are equal.*

Further study of the difference between Futures and Forward price necessitate more detailed modelling of interest rates.

5 Pricing models, Risk Neutral Probability

5.1 Risk neutral probability

In corporate finance, people usually take the "expected value of cash flows" discounted at a certain rate "corresponding to their class of risk". Under which probability should the cash flows be discounted ??

The existence of forward markets provides another way of looking at things. Consider a probability measure under which you want to take the "expected value of cash flows". This probability measure has to be able to give a zero price for any forward contract. Differently stated, it has to give a price zero to any contingent claim with 0 NPV. Equivalently, it has to be consistent with the No Arbitrage assumption.

Proposition 5 (No Arbitrage)**.** *The following propositions should hold:*

1. The price of any cash flow with zero NPV has to be equal to zero;
2. The Expected value under the "pricing measure" of an asset, should be its forward value.

We might want to the price of a -random- flow $\tilde{X}$ paid at date T with a price functional. What are the constraints of this functional. Let k a real number and Q a measure, non necessarily positive a priori.

$$Price\left(\tilde{X}_T\right) = kE^Q\left(\tilde{X}_T\right)$$

First, consider a claim which pays 1 in a specific state of the world and zero elsewhere and denote it 1_A. The price of that claim has to be strictly positive otherwise it would be a free lunch. As a result

$$\forall A : kE^Q\left(1_A\right) > 0$$

which means that Q is a positive measure. We scale Q so that it is a probability i.e. $E^Q\left(1\right) = 1$.

Now, when $A = \Omega$, $1_\Omega = 1$ $a.s.$ and

$$Price\left(1\right) = k = P\left(0,T\right)$$

Eventually, let $\tilde{X}\left(T\right)$ the price of an asset and $F\left(T\right)$ its corresponding price. By definition of the forward price

$$Price\left(\tilde{X}\left(T\right) - F\left(T\right)\right) = 0$$
$$\Leftrightarrow E^Q\left(\tilde{X}_T\right) = F\left(T\right)$$

This can be summarized in the following proposition:

Proposition 6 (**Fundamental Pricing Theorem**)**.** *In the absence of arbitrage, any claim can be priced as the discounted value of its expected returns under a probability Q under which the expected value of any asset is equal to its forward price. In particular, for any asset with price $\tilde{S}\left(T\right)$ which forward price is $F\left(T\right)$ and for any function $g : \mathbb{R} \to \mathbb{R}$ one has*

$$Price\left(g\left(\tilde{S}\left(T\right)\right)\right) = P\left(0,T\right)E^Q\left(g\left(\tilde{S}\left(T\right)\right)\right)$$

and

$$E^Q\left(S\tilde{(T)}\right) = F\left(T\right)$$

5.2 Examples

5.2.1 Stocks

Let $\tilde{S}\left(T\right)$ be the value of a stock which does not pay dividends. One has

$$E^Q\left(\tilde{S}\left(T\right)\right) = \frac{S\left(0\right)}{P\left(0,T\right)} = S_0e^{rT}$$

Now, what should be the risk neutral trend μ of an asset when this asset yields dividends ? Consider the case where a continuous dividend yield d is paid. Then, buying one unit of stock ensures to have

1. the stock value at any future value

2. a continuous flow of dividends

As a result, under the RNP

$$
\begin{aligned}
S_0 &= E\left(\int_0^t de^{-rs} S_s ds\right) + E\left(e^{-rt} S_t\right) \\
&= S_0 \times \left(d \int_0^t e^{(\mu-r)s} ds + e^{(\mu-r)t}\right) \\
&= S_0 \times \left(\frac{d}{\mu - r}\left(e^{(\mu-r)t} - 1\right) + e^{(\mu-r)t}\right)
\end{aligned}
$$

The only way for this equality to hold for any t is to ensure that

$$\mu = r - d$$

So under the RPN, a dividend paying asset trends at a rate $r - d$. Equivalently:

$$E^Q\left(S_t\right) = S_0 e^{(r-d)t}$$

5.2.2 Currencies

For a currency $X(T)$ which defines the value of a foreign currency for 1 unit of domestic currency:

$$E^Q\left(\tilde{X}(T)\right) = X(0)\frac{P^{domestic}(0,T)}{P^{foreign}(0,T)} = X_0 e^{\left(r^f - r^d\right)T}$$

5.3 The scaling of variance

We would like to model

$$\tilde{S}_T \overset{law}{=} e^{Y(T)}$$

where $Y(T)$ is a Gaussian random variable $N\left(\mu(T), \sigma^2(T)\right)$. What is a natural choice for function $\sigma^2(T)$? We choose an arbitrary subdivision of n sub-intervals of length δ/n between 0 and T $\tau_n = \{0, \delta, i\delta, ..., T\}$

$$Y(T) = \sum_{i \in \tau_n} \Delta Y_i$$

We make a first assumptions which is that the ΔY are independant

1. The ΔY_i are drawn from the same law (stationarity) with finite variance
2. They are independent

The fact that volatility is stationary means that $\sigma_i^2 = \sigma^2$. Now independence yields that

$$\sigma^2(t_1 + t_2) = \sigma^2(t_1) + \sigma^2(t_2)$$

and hence

$$
\begin{aligned}
\sigma^2(T) &= \sigma^2 T \\
\sigma(T) &= \sigma\sqrt{T}
\end{aligned}
$$

5.4 The Black and Scholes model

Theorem 1 (Gaussian Laplace Transform). *Let $X \sim N\left(\mu, \sigma^2\right)$. For any $\lambda \in \mathbb{R}$*

$$E\left(e^{\lambda X}\right) = e^{\lambda\mu + \frac{\lambda^2\sigma^2}{2}}$$

Because of Theorem 1, in the case of stock prices

$$
\begin{aligned}
Y(T) &\sim N\left(rT - \frac{1}{2}\sigma^2 T, \sigma^2 T\right) \\
\Rightarrow & E\left(S_0 e^{Y(T)}\right) = S_0 e^{rT}
\end{aligned}
$$

All the above can be obtained through the use of stochastic calculus.

The Black and Scholes model is a model where the underlying price is modelled as a log-normal diffusion

$$\begin{aligned} dS_t &= S_t \left(\mu dt + \sigma dW_t^P\right) \\ &= S_t \left(rdt + \sigma dW_t^Q\right) \end{aligned}$$

and where interest rates are constant equal to r. In such model

$$\begin{aligned} S_t &= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^P} \\ E^P\left(S_t\right) &= S_0 e^{\mu t} \\ E^Q\left(S_t\right) &= S_0 e^{rt} \end{aligned}$$

To be continued...

6 Option pricing as discounting expected cash flows

6.1 Definitions and properties

As opposed to futures and forwards, options are contracts which can possibly be exercised by the option holder.

Definition 9. *The right to buy a stock at some future date T at some predefined price K is called a "call option". The pre agreed price K is called the strike price. T is called the maturity of the option. The right to sell a stock at a predefined strike price is called a put option*

-

Definition 10. *A European option is a payoff which is paid at a certain date T, with no intermediary payment or exercise. An American option is an option that can be exercised at any moment between its first date ane maturity.*

One shall keep in mind the below useful result:

As it has been shown in (**??**), the historical probability P is not the probability under which pricing is made. Pricing is necessarily made under a probability which ensures that the expected future value of any underlying is equal to the cost of carry.

Definition 11. *A payoff is a function G which describes the amount to be paid as a function of several observables variables such as the final value of the asset, its maximum or minimum, its average, the event of having hit a barrier, etc. Typically, the payoff of call and put options can be written as*

$$\begin{aligned} Call\left(S_T, K\right) &= \left(S_T - K\right)^+ \\ Put\left(S_T, K\right) &= \left(K - S_T\right)^+ \\ with\ x^+ &= max\left(x, 0\right) \end{aligned}$$

Proposition 7 (Call put parity). *One note that*

$$\left(S_T - K\right)^+ - \left(K - S_T\right)^+ = S_T - K$$

so when no dividends are paid

$$\begin{aligned} call\left(S_0, K, T\right) - put\left(S_0, K, T\right) &= Spot - KP\left(0, T\right) \\ &= S_0 - Ke^{-rT} \end{aligned}$$

6.2 Factors affecting option prices

We consider the special case of European payoffs g which are function of S_T,a continuously dividend paying asset. The price of such asset is equal to

$$\pi\left(S_0\right) = e^{-rT} E\left(g\left(S_T\right)\right)$$

Definition 12 (Intrinsic value). *The intrinsic value of a payoff is equal to $IV = e^{-rT} g\left(E\left(S_T\right)\right)$. It is also equal to the price of the payoff in a model with no randomness*

-

Definition 13 (Time value). *The Time Value of a payoff is the difference between its price and the Intrinsic Value*

$$Price = IV + TV$$

The sign of the intrinsic value is related to the convexity of g. When g is convex, the Time Value is positive. When it is concave, the time value is negative.

To be continued

6.3 Example of strategies involving options

See several payoff examples in this https://yukalifr.files.wordpress.com/2023/10/forwardoptionpayoffs.pdf.

6.4 B&S formulas

The B&S formula without dividends

$$C = S_0 N(d_1) - K e^{-rT} N(d_2), \tag{3}$$
$$P = K e^{-rT} N(-d_2) - S_0 N(-d_1), \tag{4}$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{5}$$
$$d_2 = d_1 - \sigma\sqrt{T}. \tag{6}$$

The B&S formula with dividends

$$C = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2), \tag{7}$$
$$P = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1), \tag{8}$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \tag{9}$$
$$d_2 = d_1 - \sigma\sqrt{T}. \tag{10}$$

The B&S formula with Forward and Zero-Coupon as inputs(aka the Black Formula)

$$C = P(0,T)\left(F N(d_1) - K N(d_2)\right), \tag{11}$$
$$P = P(0,T)\left(K N(-d_2) - F N(-d_1)\right), \tag{12}$$

with

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}, \tag{13}$$
$$d_2 = d_1 - \sigma\sqrt{T}. \tag{14}$$

6.5 Options on different underlyings

Use the Black formula with forward as an input.

7 Option pricing from hedging: the Black & Scholes approach (complex)

7.1 A PDE for option prices

Seminal papers: [4] and [3].

It is assumed that the asset does not pay any dividend. Let

- S be the price process with mean μ and volatility σ
- K be the strike price of the option
- r the constant positive interest rate
- T the time to maturity
- W a brownian motion

C or $C(t, S_t)$ will be used for the Call price while P or $C(t, S_t)$ for the Put price. When necessary we might use C^a or C^e to differentiate between the American or European type of exercise.

One has: $C(T, x) = (x - K)^+$ and $P(T, x) = (K - x)^+$

The spot is a solution of the following Stochastic Differential Equation:

$$dSt = S_t(\mu dt + \sigma dW_t) \tag{15}$$

with initial value S_0.

Locally, option price variations are described by the Ito formula:

$$dC = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right] dt + \frac{\partial C}{\partial S}\sigma S dW_t$$

Now consider a portfolio comprising one option and δ stocks. Its value V varies as follows:

$$dV = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \delta \mu S\right] dt + \sigma S \left(\frac{\partial C}{\partial S} + \delta\right) dW_t$$

At each moment t, if $\delta = -\frac{\partial C}{\partial S}$ then brownian terms disappear and the portfolio bears no risk between t and $t + dt$. Accordingly, its instantaneous return must be equal to rdt because any other value would imply an arbitrage opportunity. So the price of the option satisfies a partial differential equation, which no longer depends on μ:

$$\begin{gathered}\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \\ C(T, x) = (x - K)^+\end{gathered} \tag{16}$$

Equation (16) is known as the Black and Scholes equation. How can this equation be solved ?

7.2 Solving the PDE

The PDE can be solved by usual methods such as Fourier transform or numerical methods.

The solution of a linear PDE can also be interpreted as the expectation of some function under some probabilities and hence be connected to the previous approach. This relationship between PDE and expectation is sometimes attributed to Feynman and Kac and called the Feynman-Kac approach.

Let's sketch how this works.

7.2.1 The Feynman-Kac equivalence

Define a process

$$\begin{aligned} dX(t) &= \mu(t, X)\, dt + \sigma(t, X)\, dW_t \\ X(0) &= X_0 \end{aligned} \tag{17}$$

with μ and σ satisfying Lipshitz condition so that equation (17) has a unique strong solution.

Define the corresponding infinitesimal generator

$$\mathcal{L}f\left(t,x\right)=\frac{\partial f}{\partial t}+\mu\left(t,x\right)\frac{\partial f}{\partial x}+\frac{1}{2}\sigma^{2}\left(t,x\right)\frac{\partial^{2}f}{\partial x^{2}} \tag{18}$$

For any $C^{1,2}$ function f, Ito formula yields

$$df\left(t,X\right)=\mathcal{L}f dt+\frac{\partial f}{\partial x}\sigma\left(t,X\right)dW_{t}$$

Now define

$$Z_{t}=f\left(t,X_{t}\right)\equiv\mathbb{E}\left(F\left(X_{T}\right)|\mathcal{F}_{t}\right) \tag{19}$$

By the tower property, for any $u>t$:

$$\mathbb{E}\left(F\left(X_{T}\right)|\mathcal{F}_{t}\right)=\mathbb{E}\left(\mathbb{E}\left(F\left(X_{T}\right)|\mathcal{F}_{u}\right)|\mathcal{F}_{t}\right)$$

or equivalently

$$\begin{gathered}f\left(t,X_{t}\right)=\mathbb{E}\left(f\left(u,X_{u}\right)|\mathcal{F}_{t}\right)\\ Z_{t}=\mathbb{E}\left(Z_{u}|\mathcal{F}_{t}\right)\end{gathered}$$

Define the Dirichlet problem:

$$P\left(\mathcal{L},F\right)=\begin{cases}\mathcal{L}f & =0\\ f\left(T,x\right) & =F\left(x\right)\end{cases}$$

For $f\left(t,X_{t}\right)$ to be a martingale with final value $F(X_{T})$, f should not have any finite variation part and hence $\mathcal{L}f(t,X_{t})=0$ for any possible X_{t}. This implies $\mathcal{L}f=0$. As a result, f has to be a solution of the Dirichlet problem $P\left(\mathcal{L},F\right)$.

Reciprocally, if f is a solution of the Dirichlet problem $P\left(\mathcal{L},F\right)$ then $f\left(t,X_{t}\right)$ defined by equation (19) is a martingale with final value $F(X_{T})$.

7.2.2 Application to the B&S PDE

Equation (16))does not immediately define a Dirichlet problem because of the second term rC.

To remove the second term, we introduce the forward price $F(t,x)\equiv e^{r(T-t)}C(t,x)$. The space derivatives of f and C are proportional: $\frac{\partial F}{\partial x}=e^{r(T-t)}\frac{\partial C}{\partial x}$. Oppositely, time derivatives differ because of the discounting: $\frac{\partial F}{\partial t}=e^{r(T-t)}\times(\frac{\partial C}{\partial t}-rC)$.

With that backdrop, multiplying equation (16) by $e^{r(T-t)}$ allows to obtain an equation on the forward price which we can write as:

$$P\left(\mathcal{L}^{\mathcal{BS}},(x-K)^{+}\right)=\begin{cases}\mathcal{L}^{\mathcal{BS}}f & =0\\ F\left(T,x\right) & =(x-K)^{+}\end{cases}$$

where $\mathcal{L}^{\mathcal{BS}}$ is defined as in equation 18 with $\mu(t,x)=rx$ and $\sigma(t,x)=\sigma^{2}x^{2}$.

With two other changes of variables

- Forward underlying price instead of spot underlying price: $x\rightarrow e^{r(T-t)}x$
- Log spot price instead of spot price: $x\rightarrow ln(x)$

equation (16) would read:

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^{2}\frac{\partial^{2}C}{\partial x^{2}}=0 \tag{20}$$

$$C(T,x)=(e^{x}-K)^{+} \tag{21}$$

which is known as the Heat Equation.

8 Hedging and the greeks

8.1 Definition and representation

Definition Graphical representation Convexity

8.2 Synthetic summary

Parameter	Greek	B&S (Call)	Call Option			Put Option		
			IV	TV	Price	IV	TV	Price
Delta	$\delta = \frac{\partial P}{\partial S}$	$N(d_1)$	↗	↗↘	↗	↘	↗↘	↘
Gamma	$\gamma = \frac{\partial^2 P}{\partial S^2}$	$n(d_1)$	- ↑ -	↗↘	↗↘	- ↑ -	↗↘	↗↘
Vega	$\mathcal{V} = \frac{\partial P}{\partial \sigma}$	$SN(d_1)\sqrt{\tau}$	–	↗	↗	–	↗	↗
Theta	$\Theta = \frac{\partial P}{\partial t}$		↘	↘ *	↘	↗	↘ *	↗↘
Rho	$\rho = \frac{\partial P}{\partial r}$		↗	↘	↗	↘	↘	↘
-	$\frac{\partial P}{\partial K}$	$-e^{-r\tau}N(d_2)$	↘	?	↘	?	↘	↘

Table 1: Sensitivities of call and put prices to the parameters of the Black–Scholes model. Signs indicate the direction of the price change when the parameter increases (IV = Intrinsic Value, TV = Time Value)

* Almost always true (do calculations)

8.3 Different approaches

Graphical representation PDE approach Rederivation of the B&S formula

9 Options Use Cases

9.1 Building different payoffs

See: https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_model.
Call Spread, Digital Option, straddle options

9.2 Principal Protected Products

ZC + Call, Spot + Put

9.3 Return enhancement

Selling calls, Selling puts

Appendix

Some example of reliable data sources

Source	Comments	Web url	Free API / R package
Wikipedia	A wealth of information about virtually everything.	`https://www.wikipedia.org`	Semantic Web, dbpedia, wikidata, etc. R: Glitter
Bank of International Settlements	Plenty of data about trading volumes, bank capital etc.	`https://www.bis.org`	R BISdata
World Bank	World macro economic data	`https://data.worldbank.org`	R WDI
ISO	Standardized names for countries, currencies, exchanges etc.	`http://www.iso.org`	R ISOcodes
Yahoo finance	Stock prices historical High/Low/Close and Volume data	`https://finance.yahoo.com/`	R yahoofinancer
The Guardian	A reference UK based daily newspaper	`https://www.theguardian.com/`	`https://open-platform.theguardian.com/`

Table 2: Some reliable data sources

References

[1] John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. The relation between forward prices and futures prices. *Journal of Financial Economics*, 9(4):321–346, 1981.

[2] John Hull. *Options, futures, & other derivatives. Solutions manual.* Prentice Hall International, 2006.

[3] Robert C Merton. Theory of rational option pricing. *The Bell Journal of economics and management science*, pages 141–183, 1973.

[4] Myron Scholes and Fischer Black. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.