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Interest Rates Models, Lecture Notes

MMMEF & IRFA
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Abstract This lecture borrows notations from [1]. Work in progress; made available for students only. Please do not forward. Do not quote..

Contents

1 Notations and fundamental concepts

1.1 Notation review

- instantaneous short-term rate: r_t
- Bank account: $B(t) = exp\left(\int_0^t r_s ds\right)$
- Stochastic Discount Factor: $D(t,T) = \frac{B(t)}{B(T)}$
- Zero-coupon bond: $P(t,T) = E(D(t,T) \mid \mathcal{F}_t)$; $P(T,T) = 1$.
- Day coounting conventions between as time t and T encapsulated as $\tau(t,T)$. Example of difference between act/360 and act/365.
- Continuously compunded rate: $R(t,T) := -\frac{ln(P(t,T))}{\tau}$
- Simply compounded rate, Libor rate $L(t,S)$ such that $P(t,S) = (1 + \tau L(t,S))^{-1}$
- $\mathcal{T} := \{T_{\alpha+1}, ..., T_\beta\}$, a schedule of dates.

Definition 1. *The zero-coupon curve (sometimes also referred to as "yield curve") at time t is the graph of the function*

$$T \to \begin{cases} L(t,T) & t < T \leq t+1(\textit{Short Term}) \\ Y(t,T) & T > t+1(\textit{Long Term}) \end{cases}$$

Note that money market rate (T<1) are simply compounded and others are continuously compounded.

Definition 2. *A Forward Rate Agreement (FRA) is a contract involving three time instants: The current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest-rate payment for the period between T and S. At the maturity S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $L(T,S)$ resetting in T and with maturity S. Basically, a FRA allows one to lock-in the interest rate between times T and S at a desired value K, with the rates in the contract that are simply compounded. The total value of a FRA at time S is therefore*

$$FRA(T,S) = N\tau(T,S)(K - L(T,S))$$

The pricing of the fixed leg, $N\tau K$ paid at date S is straightforward and equal to $N\tau KP(T,S)$.

The pricing of the floating leg, τL, involves a No Arbitrage reasoning. A strategy to receive the libor rate in S consists in having some have cash at T and redeem it at S which means that the NPV of $N\tau L(T,S)$ is equal to $N(P(t,T) - P(t,S))$. This is sometimes referred to as "a floating leg always quotes at par".

The level K which ensures the FRA to be fair is called the forward rate defined by:

$$\tau F(t,T,S) \times P(t,S) = P(t,T) - P(t,S) \tag{1}$$
$$F(t,T,S) := \frac{1}{\tau}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$$

Obviously, $F(T,T,S) = L(T,S)$

One can define the instantaneous forward rate: $f(t,T) := \lim_{S \to T^+} F(t,T,S) = -\frac{\partial lnP(t,T)}{\partial T}$. Using the previous formula, the zero-coupon bond can be computed/interpreted as the accumulation of forward rates:

$$P(t,T) = exp\left(-\int_t^T f(t,u)\,du\right)$$

1.2 Interest rates swaps

We have just considered a FRA, which is a particular contract whose "fairness" can be invoked to define forward rates. A generalization of the FRA is the Interest-Rate Swap (IRS). A prototypical Payer (Forward-start) InterestRate Swap (PFS) is a contract that exchanges payments between two differently indexed legs, starting from a future time instant. At every instant T_i in a prespecified set of dates $\mathcal{T} := \{T_{\alpha+1}, ..., T_\beta\}$ the fixed leg pays out the amount

$$N\tau_i K$$

, corresponding to a fixed interest rate K, a nominal value N and a year fraction τ_i between T_{i-1} and T_, whereas the floating leg pays the amount

$$N\tau_i L(T_{i-1}, T_i),$$

corresponding to the interest rate $L(T_{i-1}, T_i)$ resetting at the previous instant T_ for the maturity given by the current payment instant T_i, with T_α a given date. Clearly, the floating-leg rate resets at dates $T_\alpha, T_{\alpha+1}, ..., T_{\beta-1}$ and pays at dates $T_{\alpha+1}, ..., T_\beta$. We set $\mathcal{T} := \{T_{\alpha+1}, ..., T_\beta\}$ and $\tau := \tau_{\alpha+1}, ..., \tau_\beta$.

When the fixed leg is paid and the floating leg is received the IRS is termed **Payer IRS** (PFS), whereas in the other case we have a **Receiver IRS** (RFS).

The discounted payoffat a time $t < T_\alpha$ of a RFS can be expressed as $\sum_{i=\alpha+1}^{\beta} D(t, T_i)N\tau_i(K - L(T_{i-1}, T_i))$.

We can view this last contract as a portfolio of FRAs, we can value each FRA through formulas 1 and then add up the resulting values. We thus obtain

$$\begin{aligned} RFS(t, T, \tau, N, K) &= \sum_{i=\alpha+1}^{\beta} FRA(t, T_{i-1}, T_i, \tau_i, N, K) \\ &= \sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)(K - F(t; T_{i-1}, T_i)). \quad (2) \\ &= -NP(t, T_\alpha) + NP(t, T_\beta) + N \sum_{i=\alpha+1}^{\beta} \tau_i K P(t, T_i) \quad (3) \end{aligned}$$

The two legs of an IRS can be seen as two fundamental prototypical contracts. The fixed leg can be thought of as a coupon-bearing bond, and the floating leg can be thought of as a floating-rate note. An IRS can then be viewed as a contract for exchanging the coupon-bearing bond for the floating-rate note that are defined as

Definition 3. *1.5.3. The forward swap rate $S_{\alpha,\beta}(t)$ at time t for the sets of times T and year fractions τ is the rate in the fixed leg of the above IRS that makes the IRS a fair contract at the present time, i.e., it is the fixed rate K for which $RFS(t, T, \tau, N, K) = 0$. We easily obtain :*

$$S_{\alpha,\beta}(t) = \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)} \quad (4)$$

Let us divide both the numerator and the denominator in 4 by $P(t, T_\alpha)$ and notice that the definition of F in terms of Ps implies

$$\frac{P(t, T_k)}{P(t, T_\alpha)} = \prod_{j=\alpha+1}^{k} \frac{P(t, T_j)}{P(t, T_{j-1})} = \prod_{j=\alpha+1}^{k} \frac{1}{1 + \tau_j F_j(t)}$$

for all $k > \alpha$, where we have set $F_j(t) = F(t; T_{j-1}, T_j)$. Formula 4 can then be written in terms of forward rates as

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}$$

1.3 Options

1.3.1 Caps, floors

A **cap** is a contract that can be viewed as a payer IRS where each exchange payment is executed only if it has positive value. The cap discounted payoff is therefore given by

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)N\tau_i(L(T_{i-1}, T_i) - K)^+$$

Analogously, a **floor** is equivalent to a receiver IRS where each exchange payment is executed only if it has positive value. The floor discounted payoff is therefore given by

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)N\tau_i(K - L(T_{i-1}, T_i))^+$$

Each $D(t, T_i)N\tau_i(L(T_{i-1}, T_i) - K)^+$defines a contract that is termed **caplet**. The floorlet contracts are defined in an analogous way.

1.3.2 Swaptions

A European payer swaption is an option giving the right (and no obligation) to enter a payer IRS at a given future time, the swaption maturity. Usually the swaption maturity coincides with the first reset date of the underlying IRS. The underlying-IRS length ($T_\beta - T_\alpha$ in our notation) is called the tenor of the swaption. The payer-swaption payoff, discounted from the maturity T_α to the current time, is thus equal to

$$ND\left(t, T_\alpha\right)\left(\sum_{i=\alpha+1}^{\beta} \tau_i P(t, T_i)(K - F\left(t; T_{i-1}, T_i\right)\right)^+$$

Contrary to the cap case, this payoffcannot be decomposed in more elementary products, and this is a fundamental difference between the two main interest-rate derivatives. Indeed, we have seen that caps can be decomposed into the sum of the underlying caplets, each depending on a single forward rate. One can deal with each caplet separately, deriving results that can be finally put together to obtain results on the cap. The same, however, does not hold for swaptions. From an algebraic point of view, this is essentially due to the fact that the summation is inside the positive part operator, $(\cdots)+$, and not outside like in the cap case. Since the positive part operator is not distributive with respect to sums, but is a piece-wise linear and convex function, we have

$$\begin{aligned} &\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i \left(F(T_\alpha; T_{i-1}, T_i) - K\right)\right)^+ \\ &\leq \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\tau_i (F(T_\alpha; T_{i-1}, T_i) - K)^+ \end{aligned}$$

with no equality in general, so that the additive decomposition is not feasible. As a consequence, in order to value and manage swaptions contracts, we will need to consider the joint action of the rates involved in the contract payoff. From a mathematical point of view, this implies that, contrary to the cap case, correlation between different rates could be fundamental in handling swaptions.

2 Fundamental law of pricing

2.1 European pricing

2.1.1 Set-up

European pricing refers to pricing claims that entail a single (random) flow at date T seen from date 0. One can think about the set of all possible claims as all possible European options payoffs on all possible underlyings $\updownarrow\mathcal{X} = L^2\left(\Omega, \mathcal{F}, \mathcal{P}\right)$. To be specific, we can restrict ourselves to the single asset case where Ω is generated by all possible values of a stock with value $\tilde{S}_T$ and $L^2\left(\Omega\right)$, in that case, is just the set of all possible $f\left(\tilde{S}_T\right)$ with finite variance. It is also supposed that there is a zero-coupon bond i.e. a security which pays 1 at date T and which value today is equal to $P\left(0, T\right)$.

2.1.2 Cash and carry and the value of forward contracts

A forward contract is a contract which entails no flow at date 0 and where two parties agree to exchange $\tilde{S}_T$ (which is random) against a value which is known today which we denote F_T. At maturity, the payoff of that contract is $\tilde{S}_T - F_T$.

Remarkably, the strategy which consists in buying the stock at date 0 by borrowing an amount S_0 will provide a T value equal to $\tilde{S}_T - S_0/P\left(0, T\right)$:

1. $\tilde{S}_T$, because we can sell the stock that has been bought at date 0,

2. $-S_0/P\left(0, T\right)$ because we have to redeem the loan made to buy the stocks.

This strategy is called the “cash and carry” strategy.

Now, entering a cash-and-carry strategy and “selling” a foward contract would entail no initial price and generate a payoff at date T equal to $F_T - S_0/P\left(0, T\right)$. This strategy is an arbitrage if $F_T - S_0/P\left(0, T\right) > 0$. In the opposite case, the opposite strategy is an arbitrage.

Hence, if we assume the absence of arbitrage opportunities, the only possible value for F_T is to be equal to $S_0/P\left(0, T\right)$.

2.1.3 Pricing measure

Interestingly, the fact that forward contract have a known price creates a constraint on candidate pricing schemes. If we look for linear pricing functionals, on $\mathcal{X}$ we are looking for all element of possible element of $\mathcal{X}^*$. It is well known that all possible pricing functional can be described as

$$\pi_0\left(X_T\right) = k_T \mathbb{E}^{\mathbb{Q}_T}\left(X_T\right)$$

where k_T is a constant and $\mathbb{Q}_T$ a measure with sum 1, not necessarily positive. Now:

1. Q_T is a probability measure: since all claims which are strictly positive must have a strictly positive price then $\mathbb{Q}_T$ cannot measure negatively some sets.
2. $k_T = P\left(0, T\right)$: by definition of the price of 1 paid at date T
3. The price of forward contracts is nul: $\mathbb{E}^{\mathbb{Q}_T}\left(\tilde{S}_T\right) - S_0/P\left(0,T\right) = 0$

The consequence of the null price of forward contracts is that, under any candidate pricing measure:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_T}\left(\tilde{S}_T\right) &= S_0/P\left(0,T\right)\\ &= S_0 e^{rT}\end{aligned}$$

where r is the zero-coupon rate at date T seen from date 0.

2.2 Pricing trajectories

Let S_t be the price of an asset at date t. Let $\mathcal{T} \coloneqq \{t_0 = t, t_1, \ldots, t_n = T\}$ a series of dates, typically days. The accumulated bank account between t_i and t_j is denoted $D(t_i, t_j) \coloneqq \prod_{k=i}^{j-1} P(k, k+1)$.

As discussed previously, when only one date is involved the pricing formula comes from a one date spot cash and carry strategy consisting in buying 1 unit of stocks at date $t = t_0$ financed by borrowing up to t_1. Resulting wealth at t_1 is $S_1 - \frac{S(t_0)}{P(t_0,t_1)}$. Since the this has zero price, under a pricing measure specific to date t_1:

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}_1}\left(S(t_1) - \frac{S(t_0)}{P(t_0,t_1)}\right)\\ &\Rightarrow \mathbb{E}^{\mathbb{Q}_1}(S(t_1) \mid S(t_0)) = \frac{S(t_0)}{P(t_0,t_1)}\end{aligned} \tag{5}$$

2.2.1 Price as expectations with a stochastic discount factor

The weakness of the above formula is its dependency on t_1. We would like to obtain a pricing formula which would be valid for any possible date. The above strategy can be generalized in two ways:

1. Enter into a series of forward cash and carry strategies between t_i and t_{i+1}.
2. Run those cash and carry and strategies on a notional amount equal to $D(t_0, t_{i+1})$.

Such series of forward cash and carry strategies would yield the following stream of returns:

1. Date t_1: $D(t_0,t_1)S(t_1) - S(t_0)$ (because $B(t_0,t_1) = P(t_0,t_1)$)
2. ...
3. Date t_{i+1}: $D(t_0,t_{i+1})S(t_{i+1}) - D(t_0,t_i)S(t_i)$ (because $D(t_0,t_i) = \frac{D(t_0,t_{i+1})}{P(t_i,t_{i+1})}$)
4. Date t_n: $D(t_0,t_n)S(t_n) - D(t_0,t_{n-1})S(t_{n-1})$

Since this table of forward cash and carry contracts as a zero price, the expectation of their sum under a pricing measure which is not dependent on any specific t_i is equal to zero. Now, each intermediary terms vanish when summed and the resulting equation yields:

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}}(D(t_0,t_n)S(t_n) - S(t_0) \mid S(t_0)) = 0\\ &\Rightarrow S_{t_0} = \mathbb{E}^{\mathbb{Q}}(D(t_0,t_n)S(t_n \mid S(t_0))\end{aligned} \tag{6}$$

Both pricing equations (5) and (6) characterizes prices as expected discounted values. Yet, there is a crucial difference. The discount factor in equation (5) is known and hence outside the expectation. This is related to the fact that the cost of carry is related to the ZC price at date 0 which is known beforehand. On the opposite, in equation (6) the cost of financing of the forward cash and carry strategies is not known at inception. As a consequence, the discounting factor is stochastic and is done within the expectation.

2.2.2 Martingale formulation

The discounting factor can be reformulated as

$$D(t_0,t_n) = \prod_{k=i}^{j-1} P(k,k+1) = \prod_{k=i}^{j-1} e^{-r(i)(t_{i+1}-t_i)}$$
$$= \exp\left(\sum_{k=i}^{j-1} -r(i)(t_{i+1}-t_i)\right) \tag{7}$$

In order to obtain a forumla valid at any possible date (and not only on the subset $\mathcal{T}$) we can take the limit of equation (7) when $n \to +\infty$ and obtain:

$$D(t,T) = \lim_{n\to+\infty} D(t_0,t_n) = \exp\left(-\int_t^T r(s)ds\right)$$
$$= \frac{B(t)}{B(T)} \tag{8}$$

where $r(s)$ is the instantaneous short term rate and $B(t) := \exp\left(\int_0^t r(s)ds\right)$ the value of the corresponding bank account. Using that characterization of the Stochastic Discount Factor, equation 6 can be rewritten as

$$S_t = \mathbb{E}^{\mathbb{Q}}\left(D(t,T)S(T) \mid \mathcal{F}_t\right) = \mathbb{E}^{\mathbb{Q}}\left(\frac{B(t)}{B(T)}S(T) \mid \mathcal{F}_t\right)$$
$$\Leftrightarrow \frac{S_t}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{S(T)}{B(T)} \mid \mathcal{F}_t\right) \tag{9}$$

The previous can be summarized in the following proposition:

Proposition 1 (Fundamental Theorem of Asset Pricing)**.** *For any market where there are No Arbitrage opportunities, the following assertions hold and are equivalent*

1. *The price of a good is the expectation of its discounted value:* $S_t = \mathbb{E}^{\mathbb{Q}}\left(D(t,T)S(T) \mid \mathcal{F}_t\right)$
2. *Prices in units of bank account are martingale* $\frac{S_t}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left(\frac{S(T)}{B(T)} \mid \mathcal{F}_t\right)$

2.2.3 Consequences on the price of ZC bond

An important consequence of proposition (1) is that the price of all ZC bonds can be deduced from the dynamics of the short-term rate:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right)$$

2.2.4 Consequences on the drift of an asset under the RPN

Let $S(t)$ a candidate price process solution of an SDE with coefficient $\mu(.,.)$ and $\sigma(.,.)$.

$$d\left(\frac{S(t)}{B(t)}\right) = \frac{1}{B(t)}dS(t) - \frac{S(t)}{B^2(t)}dB(t) + d\left\langle\frac{1}{B(t)}, S(t)\right\rangle$$
$$= \frac{S(t)}{B(t)}\left(\mu(.,.)dt + \sigma(.,.)dW(t) - r(t)dt + 0\right)$$
$$= \frac{S(t)}{B(t)}\left(\left[\mu(.,.) - r(t)\right]dt + \sigma(.,.)dW(t)\right)$$

$\frac{S(t)}{B(t)}$ is a martingale iff it has no drift which implies that $\mu(.,.) - r(t) = 0 \mathbb{Q} a.s.$; As a consequence, under $\mathbb{Q}$, the drift of any possible asset is equal to the short term rate $r(s)$.

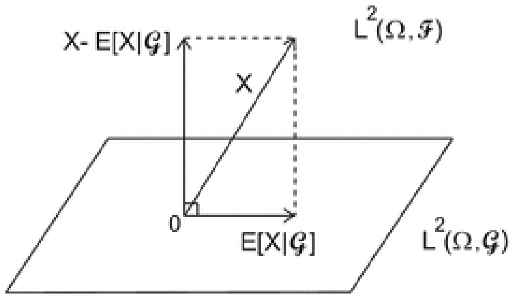

Figure 1: Conditional expectation as a projection

3 Conditional expectation, change of measures and numeraires

3.1 Conditional expectation as a projection in $\mathbb{L}^2$

L^p are Banach spaces. $F = \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is even a Hilbert space i.e. a Banach space with a scalar product

$$\langle X, Y \rangle = \mathbb{E}(XY)$$

Let $\mathcal{G} \subset \mathcal{F}$ and accordingly $G = L^2(\Omega, \mathcal{G}, \mathbb{P}) \subset F$. As F is a Hilbert space, the projection on G is uniquely defined as

$$\begin{aligned} \forall X \in F : \exists! p(X) \in G \mid \\ \forall Y \in G \ : \langle X - p(X), Y \rangle = 0 \\ \iff \forall Y \in G \ : \langle X, Y \rangle = \langle p(X), Y \rangle \end{aligned}$$

In probabilistic terms

$$p(X) = \mathbb{E}(X|\mathcal{G})$$

Now let $\mathbb{Q}$ be a measure equivalent to $\mathbb{P}$with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. One can define another scalar product

$$\begin{aligned} \langle X, Y \rangle_{\mathbb{Q}} &= E^{\mathbb{Q}}(XY) \\ &= \mathbb{E}(ZXY) \\ &= \langle ZX, Y \rangle \end{aligned}$$

One can define the projection on G w.r.t. to the $\mathbb{Q}$-scalar product. In the same manner. The two projections can be related in the following manner:

$$\begin{aligned} &\forall Y \in G : \langle X - q(X), Y \rangle_{\mathbb{Q}} = 0 \\ \iff &\forall Y \in G : \langle ZX - Zq(X), Y \rangle = 0 \\ \iff &\forall Y \in G : \langle ZX, Y \rangle = \langle Z, q(X) Y \rangle \\ \iff &\forall Y \in G : \langle p(ZX), Y \rangle = \langle p(Z) q(X), Y \rangle \end{aligned}$$

Now since $p(Z) q(X)$ is $\mathcal{G}$ measurable and the projection is unique, it is equal to the projection of ZX which yields the final result:

$$q(X) = \frac{p(ZX)}{p(Z)}.$$

Put in probabilistic terms:

Theorem 1. *Let $\mathbb{P}$ be a probability and $\mathbb{Q}$ and equivalent probability with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. One has:*

1. $\forall \mathcal{G} : \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}} = \mathbb{E}(Z \mid \mathcal{G})$
2. $\forall \mathcal{G} : \mathbb{E}^{\mathbb{Q}}(X \mid \mathcal{G}) = \frac{\mathbb{E}(XZ \mid \mathcal{G})}{\mathbb{E}(Z \mid \mathcal{G})}$

Theorem (1) is sometimes referred to as the Bayes rule. It is useful to think to about in the context of a projection.

3.2 Caracterisation of $\mathbb{Q}$-martingales

3.2.1 Change of measures

Let us now apply the previous result in the context of a a filtered space with filtration $\mathcal{F}_t$. In this context we introduce

$$\begin{aligned} Z_t &= \mathbb{E}(Z \mid \mathcal{F}_t) \\ Z_T &= Z \\ Z_0 &= 1 \end{aligned}$$

Z_t is a $\mathbb{P}$ martingale. $\mathbb{Q}$ measures are generally characterized by their density process. Since this density process has to be positive it is itself looked after under the form of a so called Doleans-Dade

$$\begin{aligned} dZ_t &= Z_t \lambda_t dW_t \\ Z_0 &= 1 \end{aligned}$$

Let $Y_t = \int_0^t \lambda_s dWs$ and $\langle Y \rangle_t = \int_0^t \lambda_s^2 ds$

$$\begin{aligned} Z_t &= exp\left(Y_t - \frac{1}{2}\langle Y \rangle_t\right) \\ &\equiv \mathcal{E}(\lambda_t) \end{aligned}$$

$\mathcal{E}(\lambda_t)$ is a convenient notation to avoid lengthy equations. When λ_t is deterministic, Y_t is a gaussian variable with zero mean and variance $\langle Y \rangle_t$.

Let X_t be a $\mathbb{Q}$-martingale. The Bayes rule yields

$$\begin{aligned} X_t &= \frac{\mathbb{E}(ZX \mid \mathcal{F}_t)}{Z_t} \\ \Longleftrightarrow X_t Z_t &= \mathbb{E}(Z_T X_T \mid \mathcal{F}_t) \end{aligned}$$

or equivalently

Theorem 2. *$X.$ is a $\mathbb{Q}$ martingale iff $X.Z.$ is a $\mathbb{P}$ martingale.*

This theorem is used often times to caracterise $\mathbb{Q}$ martingales. One typically applies Ito to XZ and equals the finite variation (dt) part to zero. Then for X to be a $\mathbb{Q}-$martingale

$$\begin{aligned} d(XZ) &= XdZ + ZdX + d\langle X, Z \rangle \\ &= XdZ + Z\left(dX + \frac{d\langle X, Z \rangle}{Z}\right) \end{aligned}$$

Since Z is a $\mathbb{P}$ martingale, we need $-\frac{d\langle X,Z \rangle}{Z}$ to be the trend of X under $\mathbb{P}$. This result can be formulated as one of the different versions of the Girsanov theorem:

Theorem 3 (Girsanov theorem)**.** *Let M a $\mathbb{P}$ martingale, Z the density of a measure $\mathbb{Q}$. Let*

$$Z_t = \mathbb{E}(Z \mid \mathcal{F}_t)$$

$$\begin{aligned} N_t &\equiv M_t - \int_0^t \frac{d\langle M, Z \rangle_s}{Z_s} \\ &= M_t - \int_0^t d\langle M, ln(Z) \rangle_s \end{aligned}$$

is a $\mathbb{Q}$-martingale.

Proof. $N_t \equiv M_t + \Lambda_t$ is a $\mathbb{Q}$ martingale iff NZ is a P martingale. We have

$$\begin{aligned} d(NZ) &= d((M + \Lambda) Z) \\ &= (M + \Lambda) dZ + ZdM + Zd\Lambda + d\langle M, Z \rangle \end{aligned}$$

NZ is a martingale iff

$$
\begin{aligned}
Zd\Lambda + d\langle M, Z\rangle &= 0 \\
\Leftrightarrow d\Lambda &= -\frac{d\langle M, Z\rangle}{Z} \\
\Leftrightarrow N_t &\equiv M_t - \int_0^t \frac{d\langle M, Z\rangle_s}{Z_s}
\end{aligned}
$$

□

Corollary 1. *Z_t being positive; there exists an adapted process λ satisfying the Novikov condition such that*

$$
\begin{aligned}
dZ_t &= Z_t \lambda_t dW_t \\
Z_0 &= 1
\end{aligned}
$$

then

$$
\begin{aligned}
N_t &= M_t - \int_0^t d\langle M, ln(Z)\rangle_s \\
&= M_t - \int_0^t \lambda_s ds
\end{aligned}
$$

3.2.2 Change of measures AND numeraire

In asset pricing, changes of measures also come with changes of numeraires so the following corrollary is often used

Corollary 2. *Let $N^1$and N^2 two numeraires and Q^1 and Q^2 two measures with density $Z = \frac{dQ_2}{dQ_1}$. If $\frac{X}{N^1}$ is a Q^1-martingale and $\frac{X}{N^2}$ is a Q^2-martingale then*

$$
Z_t = \frac{N_0^1}{N_0^2} \frac{N_t^2}{N_t^1} \tag{10}
$$

3.3 Fundamental pricing measures and numeraires

3.3.1 Forward neutral probabilities and ZC numeraire

Under the RNP, the dynamics to be modelled is $\frac{X_t}{B_t}$ while the final payoff is only related to X_T. This raises some issues since $\frac{X_T}{B_T}$ and X_T cannot be easily related. The $T-forward$ measure provides a simple and elegant way to handle this difficulty. The fundamental pricing equation **??** at date 0

$$
X_0 = \mathbb{E}\left(D(0,T) X_T \mid \mathcal{F}_0\right),
$$

suggests that prices are indeed related to the expectation of the final value of the payoff but under a different probability Q_T with density

$$
Z_T = \frac{dQ_T}{dQ} \equiv \frac{D(0,T)}{P(0,T)}
$$

Since $P(0,T) = \mathbb{E}(D(0,T))$, Z_T is always positive and has mass 1. One can check that

$$
Z_t = \frac{D(0,t)}{P(0,T)} P(t,T)
$$

and that, using theorem 1,

$$
\frac{X_t}{P(t,T)} = \mathbb{E}^T\left(\frac{X_T}{P(T,T)}\right) = \mathbb{E}^T(X_T)
$$

This show that prices expressed in numeraire P(t,T) are martingales under the $T-forward$ measure. Under this measure, the final value of $\frac{X_t}{P(t,T)}$ is equal to X_T. If one disregards the fact that it is unlikely that the volatility $\frac{X_t}{P(t,T)}$ is in general time homogeneous, a B&S model can be used for $\frac{X_t}{P(t,T)}$. This is the path followed to derive the Black formula for caplets.

3.3.2 Foreign neutral probability, currency numeraire and the quanto effect

Framework Let S^fa foreign asset denominated in a foreign currency which value in domestic currency is denoted X_t. All usual quantities with exponent f refer to foreign quantities. Typically, B_t^f is the foreign bank account and $B_t$is the local bank account.

In foreign currency $\frac{S^f}{B^f}$ is a Q^f martingale under the foreign RNP. If we make the assumption that the price of any foreign asset can always be changed without any friction, $S^f X^f$ can be seen as a local asset and hence, $\frac{S^f X^f}{B_t}$ is a Q-martingale. The density of the local probability with respect to the foreign one is thus

$$Z_t = \frac{B_t}{X_t^f B_t^f}$$

Dynamics of the exchange rate under the RPN Let $S^f = B^f$. $\frac{B^f X^f}{B_t}$ is a Q-martingale so under Q, the drift μ^X of X^f is necessarily equal to the difference of short term rates:

$$\begin{aligned} d\frac{B^f X^f}{B_t} &= \frac{B^f dX^f}{B_t} + \frac{X^f dB^f}{B_t} + X^f B^f d\left(\frac{1}{B_t}\right) \\ &= \frac{B^f X^f}{B_t}\left(\mu_t^X + r_t^f - r_t\right) dt + \frac{B^f X^f}{B_t}\sigma_X dW_t \end{aligned}$$

and hence

$$\mu_t^X = r_t - r_t^f$$

Dynamics of foreign assets Foreign assets can be modelled as

$$\begin{aligned} dS_t^f &= S_t^f\left(r_t^f dt + \sigma_S dW_t^f\right) \\ S_0^f &= 1 \end{aligned}$$

with $d\left\langle W^f, W\right\rangle = \rho dt$. Two methods can be followed.

First one, $\frac{S^f X^f}{B_t}$is a Q martingale so it has no trend under Q. Let μ^S the trend of S^funder Q :

$$\begin{aligned} d\frac{S^f X^f}{B_t} &= \frac{S^f dX^f}{B_t} + \frac{X^f dS^f}{B_t} + X^f S^f d\left(\frac{1}{B_t}\right) + \frac{1}{B_t} d\left\langle S^f, X^f\right\rangle \\ &= \frac{S^f X^f}{B_t}\left[\left(\mu_t^S + r_t - r_f^t - r_t + \rho\sigma_S\sigma_X\right) dt + \sigma_S dW_t^2 + \sigma dW_t\right] \end{aligned}$$

where W^2 is a Q-SBM. Necessarily

$$\mu_t^S = r_t^f - \rho\sigma_S\sigma_X$$

under Q.

Second one. Let

$$dW_t^2 = dW_t^f - \frac{d\left\langle W^f, Z_t\right\rangle}{Z_t}$$

is a Q-SBM.

$$\begin{aligned} dZ_t &= d\left(\frac{1}{Y_t}\right) \\ &= -Z_t\frac{dY_t}{Y_t} - Z_t\frac{d\left\langle Y\right\rangle_t}{Y_t^2} \\ &= Z_t\left(-\sigma_X^2 dt - \sigma_X dW_t\right) \end{aligned}$$

and hence

$$\frac{d\left\langle W^f, Z_t\right\rangle}{Z_t} = -\rho\sigma_X dt$$

hence $dW_t^2 = dW_t^f + \rho\sigma_X dt$ is a Q-SBM and we can rewrite

$$\begin{aligned} dS_t^f &= S_t^f\left(r_t^f dt + \sigma_S dW_t^f\right) \\ &= S_t^f\left(\left(r_t^f - \rho\sigma_S\sigma_X\right) dt + \sigma_S dW_t^2\right) \end{aligned}$$

and do proper computation under Q.

Example with the value of the Nasdaq paid in 1 year time, but in EUR Let Y_t the value of the Nasdaq paid at date T. One has

$$Y_t = N_0 e^{(r_f - r - \rho\sigma_S\sigma_X)(T-t)}$$

A positive covariance between Nasdaq and currency lowers the price. Because locally if Nasdaq goes up, the variation is unhedged. So, if at the same time, the currency goes up as well, this creates an instantaneous gain which has to be inputed into the price.

3.3.3 The Libor Swap Models

Remember that the forward swap rate has been defined as

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)}$$

Let $C_{\alpha\beta}(t) \equiv \sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)$. $C_{\alpha\beta}(t)$ is a portfolio of ZC which provides an adequate "numeraire" to value the swap rate. $S_{\alpha,\beta}(t)C_{\alpha,\beta}(t)$ is indeed a quoted asset and hence its discounted price shall be a martingale.

To be continued...

4 The Lognormal Forward-LIBOR Model (LFM)

The LFM sometimes referred to as BGM (Brace Gatarek Musiela) is a rigorous framework to account for a common practice. Before market models were introduced, there was no interest-rate dynamics compatible with either Black's formula for caps or Black's formula for swaptions. These formulas were actually based on mimicking the Black and Scholes model for stock options under some simplifying and inexact assumptions on the interest-rates distributions. The introduction of market models provided a new derivation of Black's formulas based on rigorous interest-rate dynamics.

4.1 The Black formula for caplets

4.1.1 Pricing of one caplet

A caplet on $L(T,S)$ involves the computation of the expected value of either

$$\frac{B(t)}{B(S)} \times (L(T,S) - K)^+$$

under the RNP or

$$(L(T,S) - K)^+$$

under the S-Forward probability. Remind that

$$\tau P(t,S) \times F(t,T,S) = P(t,T) - P(t,S)$$

and hence, $P(t,S)F(t,T,S)$ is equal to the difference of two ZC bonds. As such $F(t,T,S) = \frac{P(t,S)F(t,T,S)}{P(t,S)}$ is a Q^S martingale. For notational simplicity note $F_t \equiv F(t,T,S)$. One can model F as the solution of

$$dF_t = F_t \sigma_t dW_t$$
$$F_0 = F(0,T,S)$$

or equivalently

$$F_t = F(0,T,S)\,\mathcal{E}(\sigma_t)$$

In such model, under Q^S F_T is a Gaussian variable with mean $F(0,T,S)$ and average variance $v(T-t) \equiv \int_t^T \sigma^2 ds$. Let $BS(S,K,r,\sigma,T)$ the standard B&S formula. One has

$$E^S\left((L(T,S) - K)^+\right) = BS\left(F_0, K, 0, \sqrt{v}, T\right)$$

which yields the price of the corresponding caplet as

$$\boxed{caplet(T,S) = P(0,S)\,BS\left(F_0, K, 0, \sqrt{v}, T\right)}$$

It has to be noted that the uncertainty goes until T and the payment is done at date S. The time difference between, depending on the context, is sometimes called "natural". Paying at a different time involves another forward measure and hence necesitates a drift adjustment.

4.1.2 Pricing of a cap

It is market practice to price a cap with the following sum of Black's formulas (at time zero)

$$\mathrm{Cap}^{\mathrm{Black}}(N,\mathcal{T},N,K,\sigma_{\alpha,\beta}) = N\sum_{i=\alpha+1}^{\beta} P(0,T_i)BS\left(F(0,T_{i-1},T_i),K,0,\sigma_{\alpha,\beta},\tau_i\right)$$

where, the common volatility parameter $\sigma_{\alpha,\beta}$ is retrieved from market quotes.

4.2 The Lognormal-Forward-Libor

4.2.1 Payoffs that cannot be handled with Black-formulas

The problem with the Black formula is that it:

1. cannot handle the joint dynamics of two Forward Libor rate and,
2. it can neither account for the expectation of the price of a libor rate paid at a date different from its "natural" date.

Example1: ratchet payoffs Ratchet payoffs are payoffs which involve the difference between two Libor rates. Typically, one might be interested in receiving a function of

$$\Delta\left(T_{i-1},T_{i-2}\right) \equiv L\left(T_{i-1},T_i\right) - L\left(T_{i-2},T_{i-1}\right)$$

at date T_i. The problem here is that the dynamics of Δ involves the knowledge of **both** $L\left(T_{i-1},T_i\right)$ **and** $L\left(T_{i-2},T_{i-1}\right)$.

Example2: in arrear pay-offs In arrear payoffs generally involve the payment of $L\left(T,S\right)$ at a date which is not S ! It can be T or any other date but it involves the dynamics of $L\left(T,S\right)$ under a numeraire which is not $P\left(t,S\right)$.

As a matter of fact, it is necessary to handle the joint dynamics of **ALL** forward rates defined on a specific ladder.

4.2.2 Model notations and parameters

We assume a given set of dates $\mathcal{T}$ and denote $F_k\left(t\right) \equiv F\left(t,T_{k-1},T_k\right)$ the k^{th} "natural" forward LIBOR rate. We have seen before that $F_k\left(t\right)$ is a Q^k martingale where Q^k defined by its conditional density w.r.t. to Q, $P\left(t,T_k\right)$.

Under Q^k

$$dF_k\left(t\right) = F_k\left(t\right)\sigma_k\left(t\right)dZ_k\left(t\right)$$

where Z_k is a Q^k SBM. We introduce $\beta\left(t\right) = m$ if $T_{m-2} < t < T_{m-1}$ which is the maturity of the first forward which has not expired. "Reciprocally"

$$t \in \left[T_{\beta(t)-2},T_{\beta(t)-1}\right].$$

The volatility $\sigma_k\left(t\right)$ is chosen to be piecewise-constant i.e.:

$$\sigma_k\left(t\right) = \sigma_{k,\beta(t)}$$

This correlation matrix has to be specified (see discussion Brigo-Mercurio) page 210-211. As a popular example

$Vols$				
F_1	$\phi_1\psi_1$	$dead$	...	$dead$
F_2		$\phi_2\psi_2$	$dead$	$dead$
...				
F_M	$\phi_M\psi_1$			$\phi_M\psi_M$

$$d\left\langle Z_i,Z_j\right\rangle = \rho_{ij}dt$$

One can use a Black formula to price each related caplets but what about pricing **<u>all maturities in the same model ?</u>**

4.2.3 Drift adjustments

Choose one given Q^k under which F_k is a martingale. What is the dynamics of F_k under Q_i?

Let start with the case $T_k < T_i$. We have

$$\frac{dQ^i}{dQ} = \frac{D(0,T_i)}{P(0,T_i)}$$
$$\frac{dQ^k}{dQ} = \frac{D(0,T_k)}{P(0,T_k)}$$
$$\Rightarrow H_T \equiv \frac{dQ^i}{dQ^k} = \frac{P(0,T_k)}{P(0,T_i)} \times D(T_k,T_i)$$

Now we have to compute $H_t \equiv E^k(H_T \mid \mathcal{F}_t)$. For this we recognize that

$$\begin{aligned} P(t,T_i) &= E(D(t,T_i) \mid \mathcal{F}_t) \\ &= E(D(t,T_k)D(T_k,T_i) \mid \mathcal{F}_t) \\ &= P(t,T_k)E^k(D(T_k,T_i) \mid \mathcal{F}_t) \end{aligned}$$

from which we obtain that:

$$\begin{aligned} H_t &= \frac{P(0,T_k)}{P(0,T_i)}\frac{P(t,T_i)}{P(t,T_k)} \\ &= \frac{P(0,T_k)}{P(0,T_i)} \times \frac{1}{\prod_{j=k+1}^{i}(1+\tau_j F_j)} \end{aligned}$$

Now Girsanov theorem states that

$$dZ_k^i = dZ_k - m_i dt$$

is a Q^i-SBM iff

$$\begin{aligned} m_i dt &= d\langle Z_k, ln(H)\rangle_t \\ &= -d\left\langle Z_k, \sum_{j=k+1}^{i} ln(1+\tau_j F_j)\right\rangle \\ &= -\sum_{j=k+1}^{i}\frac{\tau_j}{1+\tau_j F_j} d\langle Z_k, F_j\rangle \\ &= -\sum_{j=k+1}^{i}\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\rho_{jk}\sigma_j dt \end{aligned}$$

where all variables are parameters except $F_j(t)$ which is known at date t. All can be seen as quanto adjustments.

As a result, under Q^i

$$dF_k = \sigma_k F_k \times \left(dZ_k^i + \sum_{j=k+1}^{i}\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\rho_{jk}\sigma_j dt\right)$$

where Z_k^i is a $Q^i - SBM$.

In the case when , we have

$$H_t = \frac{P(0,T_k)}{P(0,T_i)} \times \prod_{j=i+1}^{k}(1+\tau_j F_j)$$

and hence the drift is equal to $-\sum_{j=i+1}^{k}\frac{\tau_j F_j(t)}{1+\tau_j F_j(t)}\rho_{jk}\sigma_j dt$.

Note that the drift is time dependent due to the presence of $F_j(t)$ so no closed form solutions can be expected and all expectations have to be computed by Monte-Carlo simulations.

Yet, some good approximation can be obtained by freezing $F_j(t)$ to its initial value $F_j(0)$. We will see that in the following example.

4.2.4 Pricing of specific claims in LFM

Ratchet payoffs

$$
\begin{aligned}
E\left(D\left(0,T_i\right)\left[L(T_{i-1},T_i)-L(T_{i-2},T_{i-1})+X)\right]\right)^+ = \\
P\left(0,T_i\right)E^i\left[F_i(T_{i-1})-F_{i-1}(T_{i-2})-X\right]^+
\end{aligned}
$$

where all has been written under $P\left(t,T_i\right)$ numeraire and the T_i forward probability. Under this probability, using the drift adjustment formulas:

$$dF_i\left(t\right)=F_i\left(t\right)\sigma_i\left(t\right)dZ_i\left(t\right)$$

and

$$
\begin{aligned}
dF_{i-1}\left(t\right) &= F_{i-1}\left(t\right)\sigma_{i-1}\left(t\right)dZ_{i-1}\left(t\right)\\
&= F_{i-1}\left(t\right)\left[m_{i-1}dt+\sigma_{i-1}\left(t\right)dZ^i_{i-1}\left(t\right)\right]
\end{aligned}
$$

where Z_{i-1} is not a Q_i-SBM but Z^i_{i-1} is and

$$m_{i-1}\left(t\right)=-\frac{\rho_{i-1,i}\tau_i\sigma_iF_i\left(t\right)}{1+\tau_iF_i\left(t\right)}$$

At that stage, the only possible way to move forward is to envisage a Monte-Carlo simulation of $\left(F_{i-1},F_i\right).$ It is also possible to envisage an approximation of the drift which consists in "freezing" its value at date 0. The approximation is likely to be good as long as $\rho_{i-1,i}\tau_i\sigma_i$ is not too large. under this approximation:

$$
\begin{aligned}
m_{i-1}\left(t\right) &\approx m_{i-1}\left(0\right)\\
&= -\frac{\rho_{i-1,i}\tau_i\sigma_iF_i\left(0\right)}{1+\tau_iF_i\left(0\right)}
\end{aligned}
$$

and F_{i-1} also follows a log-normal dynamics. Ratchet payoffs can be approximated by spread payoffs and formula similar to Margrabe can be used.

In arrear payoffs and convexity adjustments So called "in arrear" payoffs involve the payment of $L\left(T,S\right)$ at date T instead of date S. This can be expressed as an S-payoff:

$$
\begin{aligned}
E\left(\frac{B\left(t\right)}{B\left(T\right)}\tau L\left(T,S\right)\right)E\left(\frac{B\left(t\right)}{B\left(T\right)}\left(\frac{1}{P\left(T,S\right)}-1\right)\right)\\
&= E\left(\frac{B\left(t\right)}{B\left(S\right)P\left(T,S\right)}\left(\frac{1}{P\left(T,S\right)}\right)\right)-P\left(t,T\right)\\
&= P\left(t,S\right)E^S\left(\frac{1}{P^2\left(T,S\right)}\right)-P\left(t,T\right)\\
&= P\left(t,S\right)E^S\left(\left(1+\tau F_S\left(T\right)\right)^2\right)-P\left(t,T\right)\\
&= P\left(t,S\right)-P\left(t,T\right)+P\left(t,S\right)\left(2\tau F_S\left(t\right)+\tau^2E^S\left(F^2_S\left(T\right)\right)\right)
\end{aligned}
$$

Now, F_S being log-normal, so is F^2_S and $E^S\left(F^2_S\left(T\right)\right)=F_S\left(t\right)exp\left(v^2_S\right)$ where $v_S=\int_t^S\sigma^2_u du$

5 The Vasicek model

5.1 Introduction

Historically, the objective of interest rates model was to relate the bond prices to the dynamics of the interest rates since, by NA, the price of 1 paid at T is the expected value of the bank account (equation ?? with $X_T=1$). The puzzle here is not only to price derivatives but also to provide a normative answer to the possible shapes of the yield curve.

There was a series of model developped in years 80, see Table 3.1 of [1] provides with a nice overview of different models. Before presenting the most famous We present here the most famous and infuential model developped by Vasicek, [3], we expose the solution of an affine family of SDEs which might be useful to tackle other concurrent models.

5.2 Solving the SDE

Vasicek assumed that the instantaneous spot rate under the real-world measure evolves as an Ornstein-Uhlenbeck process with constant coefficients. This does not say anything **a priori** about the dynamics under the risk neutral probability but Vasicek is looking for a model which is mean-reverting under both measures.

Under the RNP

$$dr\,(t) = k\,(\theta - r\,(t))\,\,dt + \sigma\,dW_t \tag{11}$$
$$r\,(0) = r_0 \tag{12}$$

where W_t is a SBM. The standard deviation parameter, σ, determines the volatility of the interest rate. The typical parameters θ, k and σ, together with the initial condition r_0, completely characterize the dynamics, and can be quickly characterized as follows, assuming a to be non-negative:

- θ: "long term mean level". All future trajectories of r will evolve around a mean level θ in the long run;
- k: "speed of reversion". a characterizes the velocity at which such trajectories will regroup around θ in time;
- σ: "instantaneous volatility", measures instant by instant the amplitude of randomness entering the system.

To solve the OU process one first introduces

$$\begin{aligned} X\,(t) &\equiv r\,(t) - \theta \\ dX\,(t) &= -kX\,(t)\,dt + \sigma dW_t \\ X\,(0) &= r_0 - \theta \end{aligned}$$

and then

$$\begin{aligned} Y\,(t) &= X\,(t)\,e^{kt} \\ dY\,(t) &= \sigma e^{kt} dW_t \\ Y\,(0) &= r_0 - \theta \end{aligned}$$

Define $I\,(t) \equiv e^{-kt} \int_t^t e^{ks} dW_s$. As a stochastic integral with deterministic integrand, $I(t)$ is a Gaussian process with zero mean and variance

$$V\,(t) \equiv V\,(I\,(t)) = \frac{1}{2k}\left(1 - e^{-2kt}\right)$$

One has

$$Y\,(t) = r_0 - \theta + \sigma I\,(t)$$
$$r\,(t) = \theta + (r_0 - \theta)\,e^{-kt} + \sigma I\,(t) \tag{13}$$
$$= \theta + (r_0 - \theta)\,e^{-kt} + \sigma \int_0^t e^{-k(t-s)} dW_s \tag{14}$$

which solves the SDE. The nice thing with introducing $I\,(t)$ is that it contains all the randomness of $r\,(t)$ and hence all calculations related to variance can be made on $I\,(t)$.

Thanks to equations 13 and 14. One can easily the following proposition:

$$\begin{aligned} r_t - \theta &= (r_0 - \theta)\,e^{-kt} + \sigma \int_0^t e^{-k(t-s)} dW_s \\ r_T - \theta &= (r_0 - \theta)\,e^{-kT} + \sigma \int_0^T e^{-k(T-s)} dW_s \\ &= e^{k(T-t)} \left[(r_0 - \theta)\,e^{-kt} + \sigma \int_0^t e^{-k(t-s)} dW_s \right] + \sigma \int_t^T e^{-k(T-s)} dW_s \\ &= e^{-k(T-t)}\,(r_t - \theta) + \sigma \underbrace{\int_t^T e^{-k(T-s)} dW_s}_{I(t,T)} \end{aligned}$$

Proposition 2. $r(t)$ *is a Gaussian process with the following properties*

$$r(T) = \theta + (r_t - \theta) e^{-k(T-t)} + \sigma I(t,T) \tag{15}$$
$$E(r_T|r_t) = \theta + (r_t - \theta) e^{-k(T-t)} \tag{16}$$
$$Var(r_T|r_t) = V(t,T) = \frac{\sigma^2}{2k}\left(1 - e^{-2k(T-t)}\right) \tag{17}$$
$$\gamma(t,s,u) = cov(r_s, r_u|r_t) \tag{18}$$
$$= E(I(t,s) I(t,u) | r(t)) \tag{19}$$
$$= \frac{\sigma^2}{2k} e^{-k(s+u)} \left(e^{2ku\wedge s} - e^{2kt}\right) \tag{20}$$

where $u \wedge s$, the covariance function of the brownian motion, is a short cut for $min(u,s)$.

Comments Proposition 2 allows to draw some insights on the short-term rates dynamics as modelled by a OU process.

- When $kt << 1$ i.e. $t << \frac{1}{k}$ then $Var(r_T|r_t) \approx T - t$ which shows that the process behaves like a brownian motion.
- When $kt >> 1$ i.e. $t >> \frac{1}{k}$ then $Var(r_T|r_t) \to \frac{\sigma^2}{2k}$ which shows that the process variance is capped and the return to the mean effect dominates the diffusion part.
- The initial value of the short-term rate, $r(t)$ is "forgotten" at an exponential speed and the long-term expected value of the process is equal to θ.
- r can be negative

5.3 The integrated short-term rate

The bond price $P(t,T)$ is the conditional expected value of the integrated short-term rate:

$$P(t,T) = E\left(e^{-\xi(t,T)} \mid \mathcal{F}_t\right)$$
$$\xi(t,T) = \int_t^T r(s)\, ds$$

Since $r(s)$ is a Gaussian process, $\xi(t,T)$ is also a Gaussian process. The bond price will be obtained once the expected value and variance of $\xi(t,T)$ have been computed.

The expected value calculation is a straightforward integration of equation 15:

$$\bar{\xi}(t,T) = E(\xi(t,T)|r_t) = \theta(T-t) + (r(t) - \theta) A(t,T)$$
$$A(t,T) = \frac{1}{k}\left(1 - e^{-k(T-t)}\right)$$

The computation of the variance of $R(t)$ is a little more involved:

$$\begin{aligned}
Var(\xi(t,T)|r_t) &= cov(\xi(t,T), \xi(t,T)) \\
&= E\left(\int_t^T I(t,s)\, ds \int_t^T I(t,u)\, du | r_t\right) \\
&= \int_t^T \int_t^T E(I(t,u) I(t,s) | r_t)\, duds \\
&= \int_t^T \int_t^T \gamma(t,s,u)\, dsdu \\
&= \frac{\sigma^2}{2k} \int_t^T \int_t^T e^{-k(s+u)} \left(e^{2ku\wedge s} - e^{2kt}\right) dsdu
\end{aligned}$$

The $u \wedge s$ function has to be determined. If one representes the square in which u and s evolve, s being the x-axis and u the y-axis, then $u \wedge s = s$ below the first diagonal and $u \wedge s = u$ above. The integrand being perfectly symmetric on u and s, the integral is equal to twice the value of the integration over one of the half spaces. Hence:

$$\begin{aligned} V(t,T) = Var\left(\xi(t,T)\,|r_t\right) &= \frac{\sigma^2}{k}\int_t^T \left[\int_t^u e^{-k(s+u)}\left(e^{2ks}-e^{2kt}\right)ds\right]du \\ &= \frac{\sigma^2}{k}\int_t^T e^{-ku}\left[\int_t^u \left(e^{ks}-e^{-ks}e^{2kt}\right)ds\right]du \\ &= \frac{\sigma^2}{k^2}\int_t^T e^{-ku}\left[\left(e^{ku}-e^{kt}\right)+e^{2kt}\left(e^{-ku}-e^{-kt}\right)\right]du \\ &= \frac{\sigma^2}{k^2}\int_t^T \left[1-2e^{-k(u-t)}+e^{-2k(u-t)}\right]du \end{aligned}$$

The 3 terms within this integral integrate respectively as:

1. $\frac{1}{2k}\times 2k(T-t)$
2. $\frac{1}{2k}\left(4e^{-k(T-t)}-4\right)$
3. $\frac{1}{2k}\left(1-e^{-2k(T-t)}\right)$

and hence

$$V(t,T) = \frac{\sigma^2}{2k^3}\left[2k(T-t)-3+4e^{-k(T-t)}-e^{-2k(T-t)}\right] \tag{21}$$

5.4 The Bond price as an expectation

The bond price involves the calculation of the expected value of the exponential of $\xi(t,T)$. Reminding proposition 4 in appendix, one has that

$$lnP(t,T) = -\bar{\xi}(t,T)+\frac{1}{2}V(t,T)$$

The zero-coupon rate, $R(t,T) \equiv \frac{1}{T-t}ln\left(P(t,T)\right)$ can then be obtained in a closed form manner.

Proposition 3. *In the Vasicek model, the ZC rate yields:*

$$R(t,T) = \frac{1}{T-t}\left(A(t,T)\,r(t)+B(t,T)\right) \tag{22}$$

with

$$\begin{aligned} A(t,T) &= \frac{1}{k}\left(1-e^{-k(T-t)}\right) \\ B(t,T) &= \left(\theta-\frac{\sigma^2}{2k^2}\right)\left(A(t,T)-(T-t)\right)-\frac{\sigma^2}{4k}A^2(t,T) \end{aligned}$$

Comments The above proposition makes explicit the affine relationship between the short-term rate and any zero-coupon rate. It illustrates the rigid relationship across the entire yield curve. Equation 22 show that

$$R(t,T) = f(t,r_t)$$

As a result

$$\begin{aligned} d\langle R_t\rangle &= \left(\frac{\partial f}{\partial r}\right)^2 d\langle r_t\rangle \\ d\langle R_t,r_t\rangle &= \left(\frac{\partial f}{\partial r}\right) d\langle r_t\rangle \\ \rho(R_t,r_t) &\equiv \frac{d\langle R_t,r_t\rangle}{\sqrt{d\langle R_t\rangle\, d\langle r_t\rangle}} \\ &= 1 \end{aligned}$$

the correlation of the variation of the entire yield curve are equal to 1.

Whenever the short-term rates moves by dr, the zero-coupon curve moves by $\frac{1}{T-t}A(t,T)$

5.5 A PDE approach to compute the Bond price

5.5.1 About the relationship between martingales and PDEs

Define a process

$$\begin{aligned} dX(t) &= \mu(t,X)\,dt + \sigma(t,X)\,dW(t) \\ X(0) &= X_0 \end{aligned} \tag{23}$$

with μ and σ satisfying Lipshitz condition so that X has a unique strong solution. This diffusion is characterized by its infinitesimal linear generator

$$f(t,x) \to \mathcal{L}f(t,x) = \frac{\partial f}{\partial t} + \mu(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}$$

Note that:

$$df(t,X) = \mathcal{L}f dt + \frac{\partial f}{\partial x}\sigma(t,X)\,dW_t$$

Now, consider a function F. We want to compute

$$E(F(X_T) \mid \mathcal{F}_t)$$

Define

$$Z_t \equiv E(F(X_T) \mid \mathcal{F}_t) \equiv f(t,X_t)$$

By the tower property, for any $u > t$

$$\begin{aligned} Z(t) &= E(F(X_T) \mid \mathcal{F}_t) \\ &= E(E(F(X_T) \mid \mathcal{F}_u) \mid \mathcal{F}_t) \\ &= E(Z_u \mid \mathcal{F}_t) \end{aligned}$$

$Z(t)$ is a martingale. Now, using Ito

$$Z(u) - Z(t) = \int_t^u \left[\mathcal{L}f ds + \frac{\partial f}{\partial x}\sigma(s,X)\,dW(s)\right]$$

and hence

$$E(Z(t) \mid \mathcal{F}_u) = Z(u) + E\left(\int_u^t \mathcal{L}f ds \mid \mathcal{F}_u\right)$$

So, the only way for Z to be a martingale is to ensure that

$$\forall t \leq u : E\left(\int_u^t \mathcal{L}f ds \mid \mathcal{F}_u\right) = 0$$

which can be proven to be equivalent to

$$\mathcal{L}f = 0$$

Now define the Dirichlet problem:

$$P(\mathcal{L},F) = \begin{cases} \mathcal{L}f & = 0 \\ f(T,x) & = F(x) \end{cases}$$

Clearly, for $f(t,X_t)$ to be a martingale then f has to be a solution of the Dirichlet problem $P(\mathcal{L},F)$. Reciprocally, if f is a solution of the Dirichlet problem $P(\mathcal{L},F)$ then $f(t,X_t) = E(F(X_T) \mid \mathcal{F}_t)$ is a martingale.

5.5.2 Quick reminder of general option pricing

Consider a diffusion under the RNP

$$\begin{aligned} dX(t) &= X(t)\,\mu(t,X(t))\,dt + X(t)\,\sigma(t,X)\,dW(t) \\ X(0) &= X_0 \end{aligned}$$

and the bank account

$$dB(t) = B(t)\,r(t)\,dt$$

Let $f(t,X_t)$ any possible price Necessarily

$$f(t,X(t))/B(t)$$

is a martingale. In the particular of (european) option pricing:

$$f(t,X(t)) = F(X(T))$$

Underlying drift

$$\begin{aligned} d\left(\frac{X(t)}{B(t)}\right) &= \frac{1}{B}dX - \frac{1}{B^2}dB \\ &= \frac{X}{B}\left((\mu - r)\,dt + \sigma dW\right) \end{aligned}$$

and hence

$$\mu = r\ \mathbb{P} - a.s.$$

B&S option price PDE

$$\begin{aligned} d\left(\frac{f(t, X(t))}{B(t)}\right) &= \frac{\mathcal{L}f}{B}dt - \frac{f}{B^2}dB(t) + ...dW(t) \\ &= \frac{1}{B}\left(\mathcal{L}f - fr(t)\,dt\right) + ...dW(t) \\ &= \frac{1}{B}\mathcal{L}^a(f)\,dt + ...dW(t) \end{aligned}$$

hence the PDE is

$$\begin{aligned} \mathcal{L}^a f &= 0 \\ f(T, x) &= F \end{aligned}$$

with

$$\mathcal{L}^a f = \frac{\partial f}{\partial t} + xr(t)\frac{\partial f}{\partial x} + \frac{1}{2}x^2\sigma^2(t, x)\frac{\partial^2 f}{\partial x^2} - r(t)\,f(t, x)$$

5.5.3 The bond price in the Vasicek model

Another way to compute the bond price in the Vasicek model consists in making profit of the markov property of the short term rate (see equation 15)

$$P(t, T, r_t) = E\left(e^{\int_t^T -r_u du}|r_t\right) \tag{24}$$

and calculate it as a function of r_t. Along the same lines as precedently, but with the particular dynamics of r

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{\partial P}{\partial r}k(\theta - r) + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}(t, r) &= rP(t, r) \\ P(T, r) &= 1 \end{aligned}$$

This equation is in general difficult to solve but we can use equation 3 to better understand the structure of P. Remember that equation 15 states:

$$r(u, r_t) = r_t e^{-k(u-t)} + \theta\left(1 - e^{-k(u-t)}\right) + \sigma\int_t^u e^{-k(u-s)}dW_s.$$

and hence

$$\frac{\partial r(u, r_t)}{\partial r_t} = e^{-k(u-t)}$$

and then

$$\begin{aligned} \int_t^T \frac{\partial r_u}{\partial r_t}du &= \frac{1}{k}\left(1 - e^{-k(T-t)}\right) \\ &= A(t, T) \end{aligned}$$

Now, the derivation of 24 yields

$$\begin{aligned} \frac{\partial P}{\partial r_t} &= -\frac{1}{k}\left(1 - e^{-k(T-t)}\right)P \\ &= -AP \end{aligned}$$

and hence, by integrating

$$P = C(t, T)\,e^{-A(t,T)r_t} \tag{25}$$

Now:

- $\frac{\partial P}{\partial r_t} = -AP,$
- $\frac{\partial^2 P}{\partial r_t^2} = A^2 P,$
- $\frac{\partial P}{\partial t} = \left(\frac{\partial C}{\partial t}\frac{1}{C} - \frac{\partial A}{\partial t} r\right) P$

With the assumption that P cannot be null the previous equation can be rewritten as

$$\frac{1}{C}\frac{\partial C}{\partial t} - Ak\left(\theta - r\right) + \frac{1}{2}\sigma^2 A^2 - Akr = 0$$

which can be simplified as:

$$\frac{\partial C}{\partial t} + \left(-k\theta A + \frac{\sigma^2}{2}A^2\right) C = 0$$
$$C\left(T, T\right) = 1$$

C is the solution of a backward ODE which can be integrated as

$$C\left(t, T\right) = exp\left(\int_t^T \left(-k\theta A_u + \frac{\sigma^2}{2}A_u^2\right) du\right)$$

Now

$$\begin{aligned} exp\left(\sigma^2 \int_t^T A_u^2\right) &= exp\left(\frac{\sigma^2}{k^2}\int_t^T \left(1 - 2e^{-k(T-t)} + e^{2k(T-t)}\right) dt\right) \\ &= exp\left(V\left(t, T\right)\right) \\ &= exp\left(\frac{\sigma^2}{2k^3}\left[2k\left(T - t\right) - 3 + 4e^{-k(T-t)} - e^{-2k(T-t)}\right]\right) \end{aligned}$$

and

$$\begin{aligned} exp\left(\int_t^T -k\theta A_u du\right) \times exp\left(-A\left(t, T\right) r\right) &= exp\left(-\theta\left(T - t\right) + \left(\theta - r\right) A\left(t, T\right)\right) \\ &= exp\left(-\bar{\xi}\left(t, T\right)\right) \end{aligned}$$

and the price of the bond is recovered as

$$P\left(t, T\right) = exp\left(-\bar{\xi}\left(t, T\right) + \frac{1}{2}V\left(t, T\right)\right)$$

5.6 A forward rate approach to compute the bond price

We have seen previously that the forward rate can be defined as

$$F\left(t, T, S\right) \coloneqq \frac{1}{\tau}\left(\frac{P\left(t, T\right)}{P\left(t, S\right)} - 1\right)$$

One can introduce the instantaneous forward rate: $f\left(t, T\right) \coloneqq \lim_{S \to T^+} F\left(t, T, S\right) = -\frac{\partial lnP(t,T)}{\partial T}$. Using the previous formula, the zero-coupon bond can be computed/interpreted as the accumulation of forward rates:

$$P\left(t, T\right) = exp\left(-\int_t^T f\left(t, u\right) du\right)$$

Equivalentlty

$$\frac{\partial P}{P\partial T} = -f\left(t, T\right) \tag{26}$$

We have already seen that the forward rate is a martingale under the T-forward measure. Let us prove that again. First

$$\begin{aligned} P\left(t, T\right) &= E\left(-exp\left(\int_t^T r_s ds\right) \mid \mathcal{F}_t\right) \\ \Rightarrow \frac{\partial P}{\partial T} &= E\left(-exp\left(\int_t^T r_s ds\right) r_T \mid \mathcal{F}_t\right) \\ &= P\left(t, T\right) E^T\left(-r_u | F_t\right) \end{aligned} \tag{27}$$

where the T-forward measure is the measure which density w.r.t. to the risk neutral is the normalised stochastic discount factor $Z_T = \frac{dQ_T}{dQ} \equiv \frac{exp\left(\int_0^T r_s ds\right)}{P(0,T)}$. The corresponding density process yields

$$Z_t = E\left(Z_T \mid \mathcal{F}_t\right) = P\left(t,T\right) \frac{exp\left(\int_0^t r_s ds\right)}{P\left(0,T\right)} \tag{28}$$

Now, by equalizing terms in equations (26) and (27) one has

$$f\left(t,T\right) = E^T\left(r_T | F_t\right) \tag{29}$$

where $E^T\left(.\right)$ stands for the expectation under the T-forward measure.

Equation (29) provides a third route to solve the bond price: computing the dynamics of the short term rate under the T-forward measure, computing its expected value and integrating over time.

5.6.1 Dynamics of the short-term rate under the T-forward measure

To apply Girsanov, one needs to calculate the covariation between Z_t and r_t. Differentiating equation (28) yields:

$$dZ_t = \frac{exp\left(\int_0^t r_s ds\right)}{P\left(0,T\right)} \frac{\partial P}{\partial r} dr + (...)dt$$

As in the previous subsection, we are looking for the bond price as

$$P\left(t,T\right) = exp\left(-A\left(t,T\right) r_t + ...\right)$$

and hence

$$\begin{aligned} \frac{exp\left(\int_0^t r_s ds\right)}{P\left(0,T\right)} \frac{\partial P}{\partial r} dr &= -\frac{exp\left(\int_0^t r_s ds\right)}{P\left(0,T\right)} PA\left(t,T\right) dr \\ &= -Z \times A\left(t,T\right) dr \\ &= -Z \times A\left(t,T\right) \sigma dW + ... \end{aligned}$$

with $A\left(t,T\right) = \frac{1}{k}\left(1 - e^{-k(T-t)}\right)$.

Now one has

$$\begin{aligned} \frac{d\left\langle Z,W\right\rangle}{Z} &= -A\left(t,T\right) \sigma d\left\langle W\right\rangle \\ &= -A\left(t,T\right) \sigma dt \end{aligned}$$

and, by a direct application of Girsanov, the process defined as:

$$\begin{aligned} dW^T &= dW - \frac{d\left\langle Z,W\right\rangle}{Z} \\ &= dW + A\left(t,T\right) \sigma dt \end{aligned}$$

is a T-SBM. Now, the dynamics of the short-term rates under the T-foward measure yields

$$dr_t = \left[k\left(\theta - r_t\right) - \sigma^2 A\left(t,T\right)\right] dt + \sigma dW^T$$

As the expected value of $r(T)$ the foward price $f\left(t,T,r_t\right)$ is a solution of the following ODE:

$$\begin{aligned} &\frac{\partial f}{\partial t} + kf = \theta k - \sigma^2 A\left(t,T\right) \\ &f\left(t,t,r\right) = r_t \end{aligned}$$

This is a linear ODE with a forcing term $g\left(t,T\right) = \theta k - \sigma^2 A\left(t,T\right) = \theta k - \frac{\sigma^2}{k} + \frac{\sigma^2}{k} e^{-k(T-t)}$.

.....

6 n-factor Hull-White model

Introduction Damiano-brigo :

"In this section we consider an interest-rate model where the instantaneous short-rate process is given by the sum of two correlated Gaussian factors plus a deterministic function that is properly chosen so as to exactly fit the current term structure of discount factors. The model is quite analytically tractable in that explicit formulas for discount bonds, European options on pure discount bonds, hence caps and floors, can be readily derived. Gaussian models like this G2++ model are very useful in practice, despite their unpleasant feature of the theoretical possibility of negative rates. Indeed, their analytical tractability considerably ease the task of pricing exotic products. The Gaussian distribution allows the derivation of explicit formulas for a number of non-plain-vanilla instruments and, combined with the analytical expression for zero-coupon bonds, leads to efficient and fairly fast numerical procedures for pricing any possible payoff. Also, finite spot and forward rates at a given time for any maturity and accrual conventions can be given an explicit analytical expression in terms of the short-rate factors at the relevant instant. This allows for easy propagation of the whole zero-coupon curve in terms of the two factors. Another consequence of the presence of two factors is that the actual variability of market rates is described in a better way: Among other improvements, a non-perfect correlation between rates of different maturities is introduced. This results in a more precise calibration to correlation-based products like European swaptions. These major advantages are the main reason why we devote so much attention to a two-factor Gaussian model."

For pedagogical reasons, a n-factor Hull-White model is presented even-if, in most practical applications, $n = 2$ is used.

6.1 The short-rate dynamics

The short rate is modelled as

$$r(t) = \sum_{i=1}^{n} x_i(t) + \varphi(t)$$

where

$$\begin{aligned} dx_i(t) &= -a_i x(t)\,dt + \sigma_i dW_i(t) \\ x_i(0) &= 0 \end{aligned}$$

and $d\langle W_1, W_2\rangle = \rho dt$. φ is a deterministic function such that $\varphi(0) = r(0)$.

In the Vasicek model, one had:

$$\varphi(t) = r_0 e^{-kt} + \theta\left(1 - e^{-kt}\right)$$

Important relationships:

$$\begin{aligned} x_i(t) &= \sigma_i \int_0^t e^{-a_i(t-u)} dW_i(u) \\ E\left(x_i(t) \mid \mathcal{F}_u\right) &= x_i(u)\left(1 - e^{-a_i(t-u)}\right) \\ \langle x_i(t)\rangle &= \frac{\sigma_i^2}{2a_i}\left(1 - e^{-2a_i t}\right) \\ \langle x_i(t), x_j(t)\rangle &= \frac{\rho\sigma_i\sigma_j}{a_i + a_j}\left(1 - e^{-(a_i+a_j)t}\right) \\ \gamma_{ij}(t,s,u) &= cov\left(x_i(s), x_j(u) \mid \mathcal{F}_t\right) \\ &= \frac{\rho\sigma_i\sigma_j}{a_i + a_j} e^{-a_i s} e^{-a_j u}\left(e^{(a_i+a_j)s\wedge u} - e^{(a_i+a_j)t}\right) \\ \langle r(t)\rangle &= \sum_{i=1}^{n}\langle x_i(t)\rangle + 2\sum_{i<j}\langle x_i(t), x_j(t)\rangle \end{aligned}$$

6.2 Pricing of a ZC bond

Lemma 1. *$I_i(t,T) \equiv \int_t^T x_i(u)$ is a gaussian variable with mean*

$$\begin{aligned} E\left(I_i(t,T)\right) &= M_i(t,T)\, x_i(t) \\ \textit{with } M_i(t,T) &= \frac{1 - e^{-a_i(T-t)}}{a_i} \end{aligned}$$

and

$$\langle I_i(t,T)\rangle = \frac{\sigma_i^2}{a_i^2}\left(T-t+\frac{2}{a_i}e^{-a_i(T-t)}-\frac{1}{a_i}e^{-2a_i(T-t)}-\frac{3}{2a_i}\right) \tag{30}$$

$$\langle I_i(t,T), I_j(t,T)\rangle = \frac{\rho\sigma_i\sigma_j}{a_i a_j}(T-t)+\frac{e^{-a_i(T-t)}-1}{a_i}+\frac{e^{-a_j(T-t)}-1}{a_j}-\frac{e^{-(a_j+a_i)(T-t)}-1}{a_j+a_i} \tag{31}$$

$$V(t,T) \equiv \left\langle \sum_{i=1}^{n} I_i(t,T)\right\rangle = \sum_{i=1}^{n}\langle I_i(t,T)\rangle + 2\sum_{i<j}^{n}\langle I_i(t,T), I_j(t,T)\rangle$$

Proof. Equation (30 is another version of equation (21). Equation (31) is obtained along the same lines as equation (21) but the demonstration has to be adapted a little bit. □

Theorem 4. *The price of the zero-coupon bond is equal to*

$$P(t,T) = exp\left\{-\phi(t,T) - \sum_{i=1}^{n} M_i(t,T)\, x_i(t) + \frac{1}{2}V(t,T)\right\}$$

where $\phi(t,T) \equiv \int_t^T \varphi(u)\, du$.

The nice thing with the previous expression is that it can be easily fit to the spot ZC curve since :

$$\begin{aligned} P(0,T) &= exp\left(-\phi(0,T)+\frac{1}{2}V(0,T)\right) \\ P(0,t) &= exp\left(-\phi(0,t)+\frac{1}{2}V(t,T)\right) \\ \Leftrightarrow \frac{P(0,T)}{P(0,t)} &= exp\left(-\phi(t,T)+\frac{1}{2}(V(0,T)-V(0,t))\right) \\ \Leftrightarrow exp(-\phi(t,T)) &= \frac{P(0,T)}{P(0,t)} exp\left(-\frac{1}{2}(V(0,T)-V(0,t))\right) \end{aligned}$$

Now define The price of the zero-coupon bond can be expressed for a non specified shape of the spot yield curve:

Theorem 5. *The price of the of the zero-coupon bond is equal to*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} exp\left(-\sum_{i=1}^{n} M_i(t,T)\, x_i(t) + \frac{1}{2}\Delta V(t,T)\right)$$

with $\Delta V(t,T) \equiv V(t,T) + V(0,t) - V(0,T)$.

The forward rate can be obtained easily as

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T} ln(P(t,T)) \\ &= -\frac{\partial}{\partial T} ln(A(t,T)) + \frac{\partial M_1(t,T)}{\partial T} x_1(t) + \frac{\partial M_2(t,T)}{\partial T} x_2(t) \end{aligned}$$

One find that

$$m_i(t) = \frac{\partial M_i(t,T)}{\partial T} = e^{-a_i(T-t)}$$

and

$$df(t,T) = ...dt + m_1(t,T)\,\sigma_1 dW_1(t) + m_2(t,T)\,\sigma_2 dW_2(t)$$

7 Appendix

7.1 Laplace transform of Gaussian variable

Proposition 4. *If* $Y = N(\mu, \sigma^2)$ *then*

$$\forall\lambda:\ E\left(e^{\lambda Y}\right) = e^{\lambda\mu + \frac{\lambda^2\sigma^2}{2}}$$

7.2 Solution of SDEs with affine coefficients

The following is borrowed from exercise 6.15 of [2]. Consider the following family of SDEs

$$\begin{aligned} dX_t &= [A(t)X_t + a(t)]\,dt + [\sigma_1(t)X_t + \sigma_2(t)]\,dW_t \\ X_0 &= X_0 \end{aligned} \tag{32}$$

with time dependant stochastic adapted affine coefficients satisfying a.s. bounded.
Define

$$\begin{aligned} M_t &\equiv \int_0^t \sigma_1(s)\,dW_s - \frac{1}{2}\int_0^t \sigma_1^2(s)\,ds \\ Z_t &\equiv exp\left(M_t + \int_0^t A(u)\,du\right) \end{aligned}$$

then the unique solution of equation 32 is provided by

$$X_t = Z_t\left[X_0 + \int_0^t \frac{a(u) - \sigma_1(u)\sigma_2(u)}{Z(u)}du + \int_0^t \frac{\sigma_2(u)}{Z_u}dW_u\right]$$

This can be checked as an exercise.

7.3 Feynman-Kac

Define a process

$$\begin{aligned} dX(t) &= \mu(t,X)\,dt + \sigma(t,X)\,dW_t \\ X(0) &= X_0 \end{aligned} \tag{33}$$

with μ and σ satisfying Lipshitz condition so that X has a unique strong solution. Define the corresponding infinitesimal generator

$$\mathcal{L}f(t,x) = \frac{\partial f}{\partial t} + \mu(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}$$

For any $C^{1,2}$ function f, Ito formula yields

$$df(t,X) = \mathcal{L}f dt + \frac{\partial f}{\partial x}\sigma(t,X)\,dW_t$$

Define

$$Z_t = f(t,X_t) \equiv E\left(F(X_T) \mid \mathcal{F}_t\right) \tag{34}$$

By the tower property, for any $u > t$

$$E\left(F(X_T) \mid \mathcal{F}_t\right) = E\left(E\left(F(X_T) \mid \mathcal{F}_u\right) \mid \mathcal{F}_t\right)$$

or equivalently

$$\begin{aligned} f(t,X_t) &= E\left(f(u,X_u) \mid \mathcal{F}_t\right) \\ Z_t &= E\left(Z_u \mid \mathcal{F}_t\right) \end{aligned}$$

Define the Dirichlet problem:

$$P(\mathcal{L},F) = \begin{cases} \mathcal{L}f & = 0 \\ f(T,x) & = F(x) \end{cases}$$

Clearly, for $f(t,X_t)$ to be a martingale then f has to be a solution of the Dirichlet problem $P(\mathcal{L},F)$. Reciprocally, if f is a solution of the Dirichlet problem $P(\mathcal{L},F)$ then $f(t,X_t)$ defined by equation (34) is a martingale.

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