TD2: Expectation calculations with change of measure

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1 Black and Scholes formula

Let $dS_t = S_t\left(rdt + \sigma dW_t\right)$ where W is a Q-SBM. We want to compute $E\left(S_T - K\right)^+ = \underbrace{E\left(S_T 1_{S_T \geq K}\right)}_{B} - \underbrace{E\left(K 1_{S_T \geq K}\right)}_{A}$. We note $N\left(x\right) = E\left(1_{Y \leq x}\right)$ where $Y \overset{law}{=} N\left(0,1\right)$.

1. Show that $S_t = S_0 exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \overset{law}{=} S_0 exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Y\right)$
2. Show that $A = KN\left(d_A\right)$ with $d_A = \frac{ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$
3. Let $Z_T \equiv \frac{S_T e^{-rT}}{S_0}$.
   (a) Show that Q^S defined by $\frac{dQ^S}{dQ} = Z_T$ is a probability equivalent to Q.
   (b) Show that $dZ_t = Z_t \sigma dW_t$
   (c) Show that, $dS_t = S_t\left(\left(r + \sigma^2\right)dt + \sigma dW_t^S\right)$ where W_t^S is a Q^S SBM
4. Show that $B = S_0 e^{rT} E^{Q^S}\left(1 S_{T \geq K}\right) = S_0 e^{rT} N\left(d_B\right)$ with $d_B = \frac{ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

2 Margrabe formula

Let

$$dX_t = X_t\left(rdt + \sigma_X dW_t^X\right)$$
$$dY_t = Y_t\left(rdt + \sigma_Y dW_t^Y\right)$$

with $d\left\langle X, Y\right\rangle_t = \rho dt$. We want to compute the value of the exchange option: $V = E\left(X_T - Y_T\right)^+$. To do this we rewrite $V = E\left(Y_T\left(\frac{X_T}{Y_T} - 1\right)^+\right)$

1. Show that $d\left(\frac{1}{Y_t}\right) = \frac{1}{Y_t}\left(-\left(r - \sigma_Y^2\right)dt - \sigma_Y dW_t^Y\right)$

2. Show that $d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left(\left(\sigma_Y^2 - \rho\sigma_X\sigma_Y\right)dt + \sigma_X dW_t^X - \sigma_Y dW_t^Y\right)$

3. Exercise 1: Define density $Z_T \equiv \frac{Y_T e^{-rT}}{S_0}$ and P^2 the corresponding probability

   (a) Show that $dZ_t = Z_t \sigma_Y dW_t^Y$

   (b) Show that $W_t^{2,X} = W_t^X - \rho\sigma_Y t$ and $W_t^{2,Y} = W_t^Y - \sigma_Y t$ are P^2 SBM

   (c) Show that $d\left(\frac{X_t}{Y_t}\right) = \frac{X_t}{Y_t}\left(\sigma_X dW_t^{S,X} - \sigma_Y dW_t^{S,Y}\right) = \frac{X_t}{Y_t}\sigma_{X/Y} dW^2$ with $\sigma_{X/Y}^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$ and W^2 is a $P^2 - SBM$

4. Show that $V = S_0 e^{rT} E^{P^2}(M-1)^+$ where $M \overset{law}{=} exp\left(-\frac{1}{2}\sigma_{X/Y}^2 T + \sigma_{X/Y}\sqrt{T}Y\right)$ with $Y \overset{law}{=} N(0,1)$.

5. Show that V is the BS price of an option with volatility $\sigma_{X/Y}$ and nul interest rate.

6. Application. Let X be the Nasdaq and Y be the S&P500. Suppose that their volatilities are both equal to 15%. Graph the price as a function of ρ. Comment.

3 Garman-Kholhagen

3.1 Exercise

Let S^f a foreign asset denominated in a foreign currency. X_t^f is the value of 1 unit of foreign currency in local curreny. All usual quantities with exponent f refer to foreign quantities. Typically, B_t^f is the foreign bank account and B_t is the local bank account. The value of the foreign asset in domestic currency is denoted S_t. Remind that the bank account is locally predictable i.e. has no brownian part.

1. Explain why $\frac{S^f}{B^f}$ is a Q^f martingale and why $\frac{S^f X^f}{B_t}$ is a Q-martingale.

2. Show that $Z_t = \frac{X_t^f B_t^f}{B_t}$ is the density of Q^f w.r.t. to Q.

3. Let $dX_t^f = X_t^f(\mu_X dt + \sigma_X dW_t)$.

   (a) Show that necessarily $\mu_t^X = r_t - r_t^f$.

   (b) Show that $dZ_t^{-1} = Z_t^{-1}\left(-\sigma_X^2 dt - \sigma_X dW_t\right)$

4. Let $dS_t^f = S_t^f\left(r_t^f dt + \sigma_S dW_t^f\right)$ with $d\left\langle W^f, W\right\rangle = \rho dt$.

   (a) Show that $dW_t^2 = dW_t^f + \rho\sigma_X dt$ is a Q-SBM

   (b) Show that $dS_t^f = S_t^f\left(\left(r_t^f - \rho\sigma_S\sigma_X\right)dt + \sigma_S dW_t^2\right)$

5. Application. Let N_t the value of the USD denominated Nasdaq. Let $I_t = \frac{N_t}{N_0}$.
   (a) What is the value today of receiving I_1 EUR in 1 year time ?
   (b) Compute it as a function of ρ for $r = r_f = 0$, $\sigma_N = 20\%$ and $\sigma_{USD/EUR} = 10\%$. Comment.

3.2 Solution

Q1: As a quoted foreign asset, $\frac{S^f}{B^f}$ is a Q^f martingale. Since this asset can be changed in domestic currency at any time, its chaned value is a price and hence $\frac{S^f X^f}{B_t}$ is a Q-martingale.

Q2: $\frac{S^f}{B^f}$ is a Q^f martingale iff $\frac{S^f}{B^f}Z$ is a Q martingale. Since $\frac{S^f X^f}{B_t}$ is a Q martingale. Hence, for all Q-meas S^f, $\frac{S^f X^f}{B_t}$ is a Q martingale iff $\frac{S^f}{B^f}Z$ is a Q martingale. As a result: $Z_t = \frac{X_t^f B_t^f}{B_t}$.

Q3a: Z_t is necessarily a martingale. From a financial perspective, because it is a quoted asset expressed in domestic bank account units. From a mathematical perspective, because it is a probability density.

Now Ito applied to Z yields

$$dZ = (\mu_X - r + r_f)\, dt + ...$$

Hence Z is a martingale iff the equality holds.

Q3b: Ito applied to Z^{-1}.

Q4a: Girsanov theorem states that

$$dW^2 \equiv dW_t^f - \frac{d\left\langle Z^{-1}, W^f \right\rangle_t}{Z^{-1}}$$

Using relation 3b : $\frac{d\left\langle Z^{-1}, W^f \right\rangle_t}{Z^{-1}} = -\rho\sigma_X dt$ which is the desired result.

Q4b: immediate from the previous results

4 Ito's lemma Cheatsheet

Martingale Let P^1 a probability and P^2 another equivalent one with density Z w.r.t. P^1. M is a P^1-martingale $\Longleftrightarrow$ ZM is a P^2-martingale

Girsanov Let W^1 a $P^1 - SBM$. Let $Z_T = \frac{dP^2}{dP^1}$ and $Z_t = E^1 (Z_T | F_t)$ then $W_t^2 = W_t^1 - \int_0^t \frac{d\left\langle Z, W^1 \right\rangle_s}{Z_s}$ is a $P^2 - SBM$. With Z_t solution of $dZ_t = Z_t \lambda_t dW_t$, $d\left\langle W, W^1 \right\rangle = \rho_s ds$ then $\frac{d\left\langle Z, W^1 \right\rangle_s}{Z_s} = \rho_s \lambda_s$

Differentiation Let X and Y two diffusions then: $d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\left\langle X, Y \right\rangle_t$

Ito, 1D $df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} d\left\langle X \right\rangle_t$

Ito,2D $df\left(X_t,Y_t\right)=\frac{\partial f}{\partial x}dX_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}d\left\langle X\right\rangle_t+\frac{\partial f}{\partial y}dY_t+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}d\left\langle Y\right\rangle_t+\frac{\partial^2 f}{\partial x\partial y}d\left\langle X,Y\right\rangle_t$

Ito, nD Let $W=\begin{pmatrix} W^1 \\ ... \\ W^n \end{pmatrix}$ and n dimensional SBM, $X=\begin{pmatrix} X^1 \\ ... \\ X^n \end{pmatrix}$ a diffusion with parameters $\sigma=\begin{pmatrix} \sigma_{11} & \\ ... & \\ \sigma_{1n} & \sigma_{nn} \end{pmatrix}$, $\mu=\begin{pmatrix} \mu^1 \\ ... \\ \mu^n \end{pmatrix}$ defined by

$$dX=\mu dt+\sigma dW_t$$

Let $f:R^n\to R$, $\nabla f=\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ ... \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$, $Hf=\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & ... & \frac{\partial^2 f}{\partial x_1\partial x_n} \\ & & \\ \frac{\partial^2 f}{\partial x_1\partial x_n} & & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$

then

$$\begin{aligned} df &= \nabla f' dX+\frac{1}{2}dX_t' H_f dX_t \\ &= \left(\nabla f'\mu+\frac{1}{2}tr\left(\sigma' H f\sigma\right)\right)dt+\nabla f'\sigma dW_t \end{aligned}$$