TD3: rehearsal before exam

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1 Vasicek model

The short term rate at date t is denoted $r(t)$ and is solution of the following Stochastic Differential Equation

$$\begin{aligned} dr(t) &= k(\theta - r(t))\, dt + \sigma\, dW(t) \qquad (1) \\ r(0) &= r_0 \end{aligned}$$

Exercise 1.1

Show that $r_t = r_0 e^{-kt} + \theta \left(1 - e^{-kt}\right) + \sigma \int_0^t e^{-k(t-u)} dW_u$ is a solution of equation (1)

Exercise 1.2

Prove that:

1. $\mathbb{E}(r_t) = r_0 e^{-kt} + \theta \left(1 - e^{-kt}\right)$
2. $Var(r_t) = \frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)$

Exercise 1.3

Discuss the behavior of $r(t)$ for "small" t and "large" t. Caracterize what is small and large and compare with the brownian case (i.e. $k = 0$)

Define the integrated short-term rate

$$X(t) \equiv \int_0^t r(s)\, ds$$

Exercise 1.4

Explain why X(t) is a gaussian process. Show that

$$\mathbb{E}(X(t)) = \theta t + (\theta - r_0) \frac{\left(1 - e^{-kt}\right)}{k}$$

Exercise 1.5

Note $\phi(u,v) \equiv \grave{c}ov(r_u, r_v)$. Define $y(t) \equiv \int_0^t e^{-k(t-u)} dW_u$. Show that

$$\phi(u,v) \equiv \sigma^2 \grave{\mathbb{E}}(y(u)\, y(v))$$

Then show that:

$$\phi(u,v) = \frac{\sigma^2}{2k} e^{-k(v+u)} \left(e^{2ku\wedge v} - 1\right)$$

Exercise 1.6

We want to compute $Var(X_t)$.

1. Explain why $Var(X_t) = Var\left(\int_0^t y(s)\, ds\right)$
2. Show that $Var(X_t) = \int_0^t \int_0^t \phi(u,v)\, dudv$
3. Prove that

$$Var(X(t)) = \frac{\sigma^2}{2k^3}\left(2kt - 3 + 4e^{-kt} - e^{-2kt}\right)$$

Exercise 1.7

Explain why the price of the zero-coupon is defined as

$$P(T) = exp(-X(T))$$

Thanks to the previous computation, compute the price of the zero-coupon as a function of k, σ, r_0 and t.

2 Around B&S Formula

Consider a situation where there is one riskless rate r and where the dynamics of some underlying price, under the risk neutral probability Q is governed by:

$$dS_t = S_t(ccdt + \sigma dW_t)$$
$$S_0 = 1$$

where W_t is a Standard Brownian Motion. σ is the volatility and cc the cost of carrying that underlying. Define the forward price at maturity T as $F_T \equiv E(S_T)$ under the Risk Neutral Probability

Define Y_t a process similar to S_t but with non trend:

$$dY_t = Y_t \sigma dW_t$$
$$Y_0 = 1$$

We want to calculate the price of a call option of strike K, and maturity T written on S_T in that model.

Exercise 2.1

Discuss the value of cc w.r.t. r when:

1. S is stock paying dividends
2. S is an FX rate
3. S is a commodity contract

We want to compute

$$\begin{aligned} C(\dots) &= e^{-rT} \times \mathbb{E}^Q \left(S_T - K\right)^+ \\ &= e^{-rT} \times \left(\mathbb{E}^Q \left(S_T 1_{S_T \geq K}\right) - \mathbb{E}^Q \left(K 1_{S_T \geq K}\right)\right) \end{aligned}$$

.

We define $B \equiv \mathbb{E}^Q \left(S_T 1_{S_T \geq K}\right)$, $A \equiv \mathbb{E}^Q \left(K 1_{S_T \geq K}\right)$ and $N\left(x\right) = E\left(1_{Y \leq x}\right)$ with $Y \overset{law}{=} N\left(0,1\right)$.

Exercise 2.2

1. Integrate S_T and Y_T.
2. Calculate F_T as a function of cc and T
3. Show that $S_T = F_T \times Y_T$.
4. Show that $A = KN\left(d_A\right)$ with $d_A = \frac{ln\left(\frac{F_T}{K}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$
5. Show that Q^S defined by $\frac{dQ^S}{dQ} = Y_T$ defines a probability equivalent to Q
6. Show that $W_t^S \equiv W_t - \sigma^2 t$ is a Q^S SBM
7. Show that, $dY_t = Y_t\left(\sigma^2 dt + \sigma dW_t^S\right)$
8. Show that $B = F_T \times \mathbb{E}^{Q^S}\left(1 S_{T \geq K}\right) = F_T \times N\left(d_B\right)$ with $d_B = \frac{ln\left(\frac{S_T}{K}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$

Exercise 2.3

1. Thanks to the above write $C(...)$ as a function $F_T,\ K,\ r,\ \sigma$ and T
2. Check that when $cc = r$ the usual B&S formula is recovered

3 Hull-White model

The short rate is modelled as

$$r\left(t\right)=\sum_{i=1}^{n}x_{i}\left(t\right)+\varphi\left(t\right)$$

where

$$\begin{aligned}dx_{i}\left(t\right)&=-a_{i}x\left(t\right)dt+\sigma_{i}dW_{i}\left(t\right)\\x_{i}\left(0\right)&=0\end{aligned}$$

and $d\left\langle W_{1},W_{2}\right\rangle=\rho dt$. φ is a deterministic function such that $\varphi\left(0\right)=r\left(0\right)$. In the Vasicek model, one had:

$$\varphi\left(t\right)=r_{0}e^{-kt}+\theta\left(1-e^{-kt}\right)$$

Exercise 3.1

Demonstrate the following relationships:

$$\begin{aligned}x_{i}\left(t\right)&=\sigma_{i}\int_{0}^{t}e^{-a_{i}\left(t-u\right)}dW_{i}\left(u\right)\\E\left(x_{i}\left(t\right)\mid\mathcal{F}_{u}\right)&=x_{i}\left(u\right)\left(1-e^{-a_{i}\left(t-u\right)}\right)\\\left\langle x_{i}\left(t\right)\right\rangle&=\frac{\sigma_{i}^{2}}{2a_{i}}\left(1-e^{-2a_{i}t}\right)\\\left\langle x_{i}\left(t\right),x_{j}\left(t\right)\right\rangle&=\frac{\rho\sigma_{i}\sigma_{j}}{a_{i}+a_{j}}\left(1-e^{-\left(a_{i}+a_{j}\right)t}\right)\\\gamma_{ij}\left(t,s,u\right)&=cov\left(x_{i}\left(s\right),x_{j}\left(u\right)\mid\mathcal{F}_{t}\right)\\&=\frac{\rho\sigma_{i}\sigma_{j}}{a_{i}+a_{j}}e^{-a_{i}s}e^{-a_{j}u}\left(e^{\left(a_{i}+a_{j}\right)s\wedge u}-e^{\left(a_{i}+a_{j}\right)t}\right)\\\left\langle r\left(t\right)\right\rangle&=\sum_{i=1}^{n}\left\langle x_{i}\left(t\right)\right\rangle+2\sum_{i<j}^{n}\left\langle x_{i}\left(t\right),x_{j}\left(t\right)\right\rangle\end{aligned}$$

Exercise 3.2

Let $I_{i}\left(t,T\right)\equiv\int_{t}^{T}x_{i}\left(u\right)$ a Gaussian variable; Show that:

$$\begin{aligned}E\left(I_{i}\left(t,T\right)\right)&=M_{i}\left(t,T\right)x_{i}\left(t\right)\\\text{with }M_{i}\left(t,T\right)&=\frac{1-e^{-a_{i}\left(T-t\right)}}{a_{i}}\end{aligned}$$

and

$$\langle I_i(t,T)\rangle = \frac{\sigma_i^2}{a_i^2}\left(T-t+\frac{2}{a_i}e^{-a_i(T-t)}-\frac{1}{a_i}e^{-2a_i(T-t)}-\frac{3}{2a_i}\right) \tag{2}$$

$$\langle I_i(t,T), I_j(t,T)\rangle = \frac{\rho\sigma_i\sigma_j}{a_i a_j}(T-t)+\frac{e^{-a_i(T-t)}-1}{a_i}+\frac{e^{-a_j(T-t)}-1}{a_j}-\frac{e^{-(a_j+a_i)(T-t)}-1}{a_j+a_i} \tag{3}$$

$$V(t,T) \equiv \left\langle \sum_{i=1}^{n} I_i(t,T)\right\rangle = \sum_{i=1}^{n}\langle I_i(t,T)\rangle + 2\sum_{i<j}^{n}\langle I_i(t,T), I_j(t,T)\rangle$$

Exercise 3.3

Show that the price of the zero-coupon bond is equal to

$$P(t,T) = exp\left\{-\phi(t,T)-\sum_{i=1}^{n} M_i(t,T)\,x_i(t)+\frac{1}{2}V(t,T)\right\}$$

where $\phi(t,T) \equiv \int_t^T \varphi(u)\,du$.

Then show that ϕ is characterized by the following relationship between spot quantities. Comment.

$$\Leftrightarrow exp(-\phi(t,T)) = \frac{P(0,T)}{P(0,t)}exp\left(-\frac{1}{2}(V(0,T)-V(0,t))\right)$$

Eventually prove that:

$$P(t,T) = \frac{P(0,T)}{P(0,t)}exp\left(-\sum_{i=1}^{n} M_i(t,T)\,x_i(t)+\frac{1}{2}\Delta V(t,T)\right)$$

with $\Delta V(t,T) \equiv V(t,T)+V(0,t)-V(0,T)$.