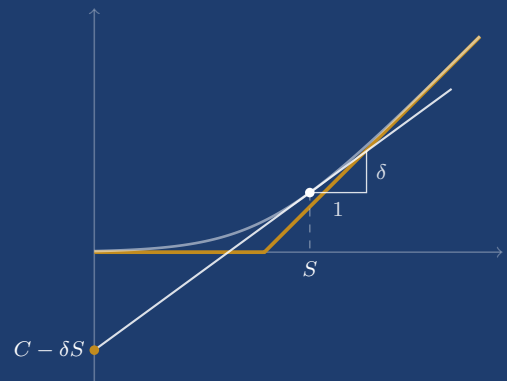


Financial Products

and Introduction to Pricing

LECTURE NOTES & EXERCISES



Master MMMEF & IRFA
Prof. Nicolas Gaussel
Academic Year 2026–2027

Contents

1	General concepts of finance and accounting	3
1.1	Quick introduction to financial markets	3
1.1.1	Those who need money	3
1.1.2	Those who save money	3
1.1.3	Intermediaries	3
1.2	Double entry accounting system	4
1.3	Joint dynamics of Assets, Equity and Debts	4
1.4	Limited liability	5
2	Fixed Income and Currency (FIC) forward contracts	5
2.1	Linear vs continuous rates	5
2.2	Compounding/discounting	6
2.3	Forward Rate Agreements	6
2.4	Forward Exchange Rate	7
2.5	Exercises	8
3	Equity forward contract	10
3.1	One date Cash and Carry arbitrage	10
3.2	Intermediate dividends	10
3.3	Exercises	11
4	Futures contracts	11
4.1	Introduction	11
4.2	Characterisation of the futures price	11
4.2.1	Characterisation of forward prices	12
4.2.2	Extension to futures prices	12
4.3	Exercises	13
5	Pricing models, Risk Neutral Probability	13
5.1	Risk neutral probability	13
5.2	Examples	14
5.2.1	Stocks	14
5.2.2	Currencies	16
5.3	The scaling of variance	16
5.4	The Black and Scholes model	17
6	Option pricing as discounting expected cash flows	17
6.1	Definitions and properties	17
6.2	Factors affecting option prices	18
6.3	Example of strategies involving options	18
6.4	B&S formulas	18
6.5	Options on different underlyings	19
6.6	Exercises	19
7	Option pricing from hedging: the Black & Scholes approach (complex)	23
7.1	A PDE for option prices	23
7.2	Solving the PDE	23
7.2.1	The Feynman-Kac equivalence	23
7.2.2	Application to the B&S PDE	24
8	Hedging and the greeks	25
8.1	Definition and representation	25
8.2	Synthetic summary	26
8.3	Different approaches	26
8.4	Exercises	26

9 Options Use Cases	27
9.1 Building different payoffs	27
9.2 Principal Protected Products	27
9.3 Return enhancement	27
9.4 Exercises	27
Solutions	28

How to use these notes. Exercises are gathered at the end of each section and their solutions at the end of the document. Unless otherwise specified, interest rates are continuously compounded and expressed per annum: the discount factor for a rate r and a maturity T is e^{-rT} .

1 General concepts of finance and accounting

1.1 Quick introduction to financial markets

Financial markets are the places where those who need money meet those who save money. This match is not straightforward: the needs of borrowers (amounts, maturities, risks) rarely coincide with the wishes of savers. Several families of market participants are involved. Orders of magnitude below are given in trillions of US dollars (Tn).

1.1.1 Those who need money

Mostly states, corporations and households:

- World equity market capitalisation: ~ 100 Tn
- World corporate debt: ~ 90 Tn
- World sovereign debt: ~ 90 Tn
- Households: mostly “contractual” loans such as mortgages, part of which are securitised. For instance, the US Mortgage Backed Securities market amounts to ~ 12 Tn.

In total, roughly 3×10^{14} USD.

1.1.2 Those who save money

Savings are mostly channelled through institutional investors:

- Asset managers: ~ 120 Tn (half of which in equity funds)
- Insurance companies: ~ 40 Tn
- Sovereign funds: ~ 10 Tn
- Endowments and foundations: ~ 2 Tn

1.1.3 Intermediaries

Between the two sides stand intermediaries (banks, brokers, asset managers, etc.) which collect savings, provide financing and transform maturities and risks.

The added value of financial markets Compared with bilateral arrangements, organised markets bring:

- Liquidity: positions can be bought or sold quickly, at a price known to all;
- Transparency: prices are public and issuers are subject to disclosure rules;
- Normalisation: standard contracts are easier to compare and to trade;
- Risk transfer: risks can be split and transferred to those willing to bear them.

Notion of “security” A security is a financial contract which has been standardised so that it can be transferred from one holder to another (shares, bonds, etc.). Financial markets have evolved from bespoke bilateral contracts towards such normalised, transferable securities.

1.2 Double entry accounting system

In the double-entry accounting system, each financial transaction is recorded by at least two accounting entries. The balance sheet of an entity shows on one side what it owns (the assets) and on the other side how these assets are financed (the liabilities, made of equity and debt).

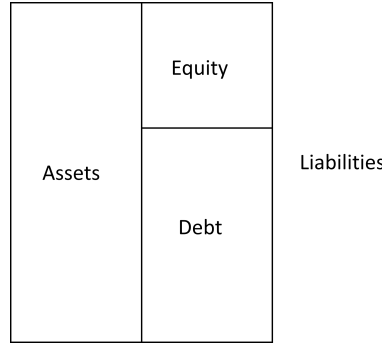


Figure 1: Asset and Liability representation

Proposition 1. *At any moment in time, for any balance sheet:*

$$Assets = Liabilities = Equity + Debt.$$

1.3 Joint dynamics of Assets, Equity and Debts

Let A_t, E_t, D_t be the respective levels of assets, equity and debt at time t , so that $A_t = E_t + D_t$.

Definition 1 (Leverage). *The ratio between assets and equity is called **leverage**: $l_t \equiv \frac{A_t}{E_t}$.*

Between t and $t + \Delta t$, the balance sheet identity imposes that

$$\Delta A_t = \Delta E_t + \Delta D_t.$$

The relative variation of equity, i.e. of the wealth of the shareholders, can then be written as

$$\begin{aligned} \frac{\Delta E_t}{E_t} &= \frac{\Delta A_t}{E_t} - \frac{\Delta D_t}{E_t} \\ &= \frac{A_t}{E_t} \frac{\Delta A_t}{A_t} - \frac{D_t}{E_t} \frac{\Delta D_t}{D_t} \\ &= l_t \frac{\Delta A_t}{A_t} - (l_t - 1) \frac{\Delta D_t}{D_t} \end{aligned}$$

or equivalently

$$\frac{\Delta E_t}{E_t} - \frac{\Delta D_t}{D_t} = l_t \left(\frac{\Delta A_t}{A_t} - \frac{\Delta D_t}{D_t} \right).$$

This accounting identity shows that the excess return of equity over debt is equal to the leverage times the excess return of assets over debt.

Let us denote $\tilde{\pi}_E \equiv \frac{\Delta E_t}{E_t} - \frac{\Delta D_t}{D_t}$ and $\tilde{\pi}_A \equiv \frac{\Delta A_t}{A_t} - \frac{\Delta D_t}{D_t}$ these excess returns. If the leverage is kept constant, equal to l , and the excess returns are modelled as stationary random variables, the following linear relationship holds:

$$\tilde{\pi}_E = l \times \tilde{\pi}_A.$$

As a result, for the expectation and the standard deviation:

$$\pi_E \equiv \mathbb{E}(\tilde{\pi}_E) = l\pi_A, \quad \sigma_E \equiv \text{stdev}(\tilde{\pi}_E) = l\sigma_A.$$

Definition 2. *The extra return of an asset above the cost of financing is called the **risk premium** (or excess return) of that asset. Its (forward looking) average is called the “**Expected Risk Premium**” or “**Expected Excess Return**”. The standard deviation of the return of an asset is called its **volatility**.*

The above equations entail the following consequences:

1. An asset whose return is lower than the return of debt has a negative risk premium.

2. The expected risk premium as well as the volatility of equity are those of the assets, linearly amplified by the leverage.

Remark 1. *In finance, stock quantities (prices, balance sheet items) are expressed in currency units. Relative variations such as π are expressed per unit of time (per annum, per day, etc.): their dimension is T^{-1} . Hence, one has to be careful about the time unit which has been chosen. In many instances, the “hidden” time unit is the year: when interest rates are said to be equal to 5%, it implicitly means “5% per year”. With that backdrop, a day is equal to $1/256$ or $1/365$ of a year, or something else depending on the convention.*

1.4 Limited liability

To foster business development, the principle of limited liability has emerged: shareholders cannot lose more than what they have invested, so that **the value of equity always remains positive**. When the company can no longer pay its bills, or when the value of its assets falls below the value of its debts, it is said to be in bankruptcy. Since equity cannot become negative, its value is floored at 0, and the losses beyond that point are borne by the debt holders.

We will come back later to the impact of this floor on the valuation of equity and debt. We can already relate leverage and bankruptcy risk: what variation of the assets wipes out the equity? Neglecting the return of debt, equity is wiped out when

$$\begin{aligned}\pi_E &\leq -100\% \\ \iff l\pi_A &\leq -1 \\ \iff \pi_A &\leq -\frac{1}{l}\end{aligned}$$

The higher the leverage, the smaller the fall of the assets which leads to bankruptcy.

Example 1 (Lombard loan). *An investor owning 100 borrows 300 through a Lombard loan (a loan collateralised by a portfolio of securities) and buys 400 of stocks. Its leverage is $l = 400/100 = 4$. Neglecting interest, a fall of the stocks of $-\frac{1}{4} = -25\%$ brings the portfolio down to 300, the amount of the loan: the equity of the investor is wiped out. The position can withstand a fall of at most 25%.*

2 Fixed Income and Currency (FIC) forward contracts

2.1 Linear vs continuous rates

In this section, the interest rate is assumed to be constant, equal to r . **Interest rates are almost always expressed per annum**, so that they can be compared with one another.

A flow of money is defined by a currency and a date of payment. While it seems obvious that $1 \text{ USD} \neq 1 \text{ EUR}$, it is generally less obvious that $1 \text{ USD today} \neq 1 \text{ USD tomorrow}$. Think about the promise of principal protected products: is it really an appealing deal?

Foreign exchange rates are the adjustment factors between the values of one unit of two different currencies. **Interest rates** are the adjustment factors between the values of one unit of the same currency paid at two different dates. Converting a value from one date to another is called **compounding** when we move forward in time and **discounting** when we move backward in time. Since deposits generally earn interest, the compounding factor is generally greater than 1 and the discounting factor lower than 1.

Let us now give a precise meaning to the above. If $r = 5\%$, what is the value in 2 years of 1 invested today? If compounding is linear, the compounding factor over a period t is

$$C = 1 + rt.$$

But it could also be agreed that interest is compounded twice during the period, in which case

$$C = \left(1 + \frac{rt}{2}\right) \times \left(1 + \frac{rt}{2}\right),$$

and, more generally, if the period is divided into n sub-periods,

$$C(n) = \left(1 + \frac{rt}{n}\right)^n.$$

The relationship between the compounding factor and the level of the interest rate therefore only makes sense once a compounding frequency has been agreed upon. This choice is not innocuous, because the compounding frequency is the smallest period over which compounding can consistently be computed: if a monthly compounding has been agreed upon, how should one compound over a week?

An answer is to consider the limit of continuous compounding, i.e. $n \rightarrow +\infty$:

$$C(t, n) = \left(1 + \frac{rt}{n}\right)^n$$

$$C(t) = \lim_{n \rightarrow \infty} C(t, n) = e^{rt}$$

Discounting being the inverse operation of compounding, the value, seen from 0, of a flow of 1 paid at date t is $B_t = C_t^{-1} = e^{-rt}$. Compounding and discounting factors can be interpreted as **exchange rates between two dates**.

In the actuarial literature, one generally finds $B_t = (1 + r_a)^{-t}$, where r_a stands for the actuarial (annually compounded) rate. The exponential form is clearer and more tractable, but both definitions are equivalent, with $r_a = e^r - 1$.

2.2 Compounding/discounting

Definition 3 (Zero Coupon Bond). *A security which pays 1 at a given date is called a zero-coupon bond. The value at date t of the zero-coupon bond paying 1 at date T is denoted $P(t, T)$.*

Definition 4 (Net Present Value). *Consider a stream of cash-flows X_τ paid at dates $\tau \equiv \{t_1, \dots, t_n\}$. The Net Present Value of X_τ is defined by*

$$NPV(t) = \sum_i P(t, t_i) \times X_i.$$

This NPV can be “transported” to any other date t_k :

$$NPV(t_k) = NPV(t) \times \frac{1}{P(t, t_k)}.$$

Definition 5 (Duration). *The duration of a security is the opposite of the relative sensitivity of its price to the level of interest rates. For a zero-coupon bond with continuously compounded rate r , $P(t, r) = e^{-rt}$ and*

$$\frac{\partial P(t, r)}{\partial r} = -tP(t, r)$$

$$\Leftrightarrow \frac{\partial P(t, r)}{P(t, r) \partial r} = -t$$

so that the duration of a zero-coupon bond is equal to its maturity t : a variation of the rate today impacts the discounting over the entire life of the bond.

Example 2 (Bullet bond). *Compute the value of a bond which pays a coupon at rate r^* every year and redeems all the capital at maturity n , when interest rates are equal to r :*

$$B(r^*, r) = r^* \sum_{i=1}^n \frac{1}{(1+r)^i} + \frac{1}{(1+r)^n} \quad (1)$$

$$= \frac{r^*}{r} \left(1 - \frac{1}{(1+r)^n}\right) + \frac{1}{(1+r)^n} \quad (2)$$

One notes that:

- $r^* > r \Rightarrow B(r^*, r) > 1$ (“above par”);
- $r^* < r \Rightarrow B(r^*, r) < 1$ (“below par”);
- $r^* = r \Rightarrow B(r^*, r) = 1$ (“at par”).

2.3 Forward Rate Agreements

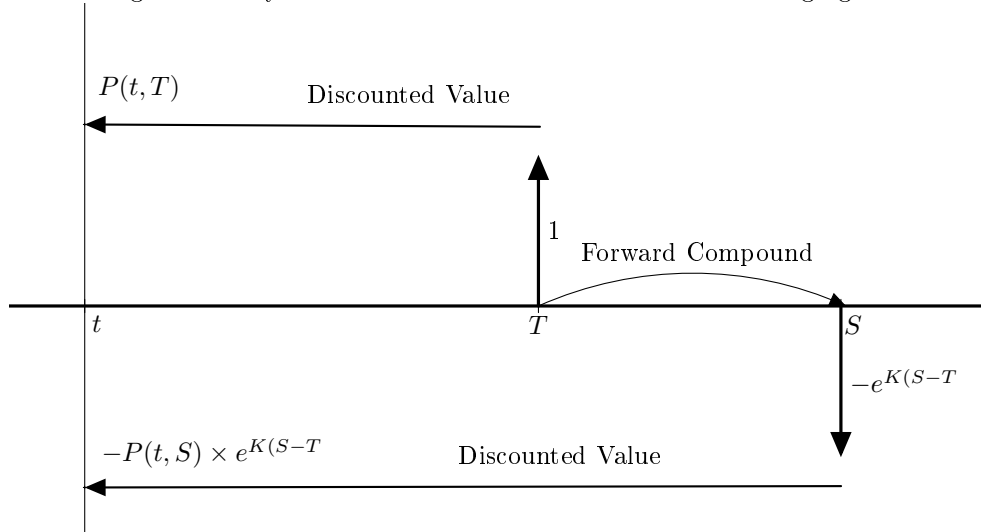
Let $R(t, T) := -\frac{\ln(P(t, T))}{T-t}$ be the continuously compounded zero-coupon rate and $L(t, T)$ the linearly compounded rate, defined by $P(t, T) = (1 + (T-t)L(t, T))^{-1}$.

Definition 6. *The zero-coupon curve (sometimes also referred to as “yield curve”) at time t is the graph of the function*

$$T \rightarrow \begin{cases} L(t, T) & t < T \leq t+1 \text{ (short term)} \\ R(t, T) & T > t+1 \text{ (long term)} \end{cases}$$

Definition 7. A *Forward Rate Agreement (FRA)* is a contract involving three dates: the current date t , the expiry date $T > t$ and the maturity date $S > T$. It allows to lock today the interest rate which will apply between T and S . At maturity S , a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $R(T, S)$, which is fixed (“resets”) at T for the maturity S .

The flows generated by this contract are summarised in the following figure:



Since this contract does not entail any initial flow, necessarily

$$P(t, T) = P(t, S) e^{K(S-T)}$$

or equivalently

$$F(t, T, S) \equiv K = \frac{1}{S - T} \ln \left(\frac{P(t, T)}{P(t, S)} \right).$$

$F(t, T, S)$ is called the continuously compounded **forward rate**. It is such that

$$R(t, S) = \frac{R(t, T)(T - t) + F(t, T, S)(S - T)}{S - t} :$$

the zero-coupon rate until S is the average of the zero-coupon rate until T and of the forward rate between T and S , weighted by the lengths of the periods.

One can also define the **instantaneous forward rate**:

$$f(t, T) := \lim_{S \rightarrow T^+} F(t, T, S) = -\frac{\partial \ln P(t, T)}{\partial T}.$$

Integrating this relation, the zero-coupon bond can be computed (and interpreted) as the accumulation of instantaneous forward rates:

$$P(t, T) = \exp \left(- \int_t^T f(t, u) du \right).$$

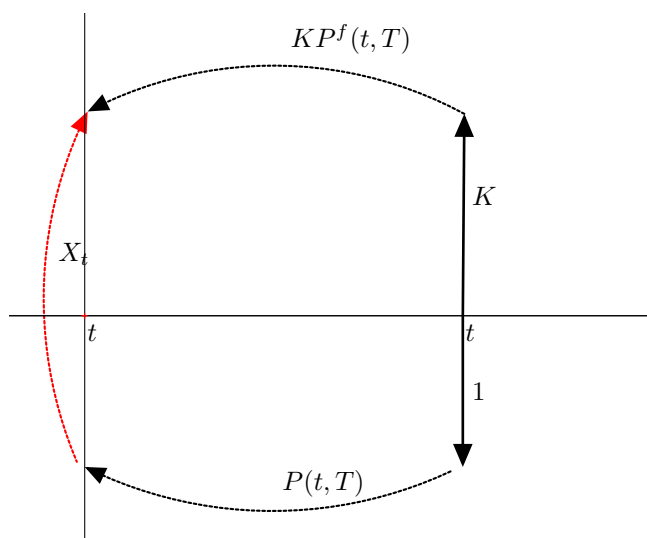
Example 3. Flat rate curve. If $R(t, T) = r$ for all T , then $F(t, T, S) = f(t, T) = r$: forward rates are equal to spot rates.

The forward as weighted average between two rates. If $R(0, 1) = 2\%$ and $R(0, 2) = 3\%$, then $F(0, 1, 2) = \frac{3\% \times 2 - 2\% \times 1}{1} = 4\%$. When the curve is upward sloping, forward rates are above zero-coupon rates.

2.4 Forward Exchange Rate

We keep the same notation and denote with a superscript f the foreign quantities. For instance, $P^f(t, T)$ is the foreign zero-coupon bond. Let X_t be the exchange rate between the two currencies, meaning that 1 unit of local currency is worth X_t units of foreign currency at date t . If X increases, the local currency appreciates; if it decreases, it depreciates.

A forward exchange rate agreement allows to lock today the exchange rate which will apply at a future date T .



By the law of one price:

$$\begin{aligned} K(t, T) &= X_t \frac{P(t, T)}{P^f(t, T)} \\ &= X_t e^{(R^f(t, T) - R(t, T))(T - t)} \end{aligned}$$

So the forward value of the local currency is above its spot value (the local currency trades at a **forward premium**) when the foreign rate is higher than the local rate, and below it (**forward discount**) otherwise. This is not a forecast of appreciation: the forward rate merely offsets the interest rate differential between the two currencies.

2.5 Exercises

Exercise 2.1

The exchange rate of US dollars is today 8.50 SEK per dollar. The forward price of a dollar to be delivered in six months is 8.40 SEK. If the Swedish six month interest rate is 4%, determine the US six month interest rate.

Exercise 2.2

We are August 1st. The forward price of a US dollar to be delivered on December 31st is 0.94630 Euros. The forward price of a US dollar to be delivered at the end of June next year is 0.95152 Euros. Assuming a flat term structure in both countries and that the Euro interest rate equals 4%, what is the level of US rates?

Exercise 2.3

Boeing just signed a contract to sell a 777 aircraft to Air France. Air France will be billed \$300 million, payable in one year. Air France has the major part of its cash in euros. The current spot exchange rate is 1.090\$/€ and the one-year forward rate is 1.112\$/€. The annual interest rate available to Air France is 5.0% in the U.S. and 3.0% in Europe. Air France is concerned with the volatile exchange rate between the dollar and the euro and would like to hedge its currency exposure.

- Describe qualitatively the risk which Air France is taking.
- Air France is considering a first hedging alternative, which would consist of buying forward the amount of USD to be paid, thanks to a currency forward contract. What would be the final euro value of such an operation?
- The second hedging alternative would consist of borrowing euros, changing them spot into USD, and lending these USD so that the amount received in 1 year is exactly \$300M. What would be the final euro value of such an operation?

Exercise 2.4 Problem: multi-period / multi-currency financing

In the following, $r^{eur}(T) = r^{eur}(0, T)$ stands for the continuously compounded euro zero-coupon rate between today and date T . The corresponding discount factor is $P^{eur}(0, T) = e^{-r^{eur}(T)T}$. The yield curve is displayed

in Table 1. The (unknown) future zero-coupon rate which will prevail at date t for maturity T is denoted $r^{eur}(t, T)$, and the (known) forward rate between dates t and T is denoted $f^{eur}(0, t, T) = f^{eur}(t, T)$.

T	$r^{eur}(T)$
1	3%
2	3.3%
3	3.6%
4	3.7%

Table 1: ZC yield curve

A friend of yours needs to borrow 100 000 € for the coming 4 years. He contemplates three possible strategies:

- Strategy SA: contract right now a 4 year ZC loan;
- Strategy SB: borrow money now for the first two years and then, in 2 years time, enter into a new 2 year loan;
- Strategy SC: borrow money now for the first three years and, in 3 years time, enter into a 1 year loan.

He asks you for some advice.

- Compute the NPV of each strategy as a function of the different rates.
- What is the level of the future ZC rate which makes strategy SB winning or losing as compared to SA? Comment.
- Same question for SC.
- Eventually, the yield curve remained unchanged during the period. Which strategy has eventually been the most interesting? Discuss.
- Propose a Python or an R code which could fill the forward rates matrix of Table 2 based on the figures of Table 1:

	$T - t = 1$	$T - t = 2$	$T - t = 3$	$T - t = 4$
$t = 0$				
$t = 1$				NA
$t = 2$			NA	NA
$t = 3$		NA	NA	NA

Table 2: Forward rate matrix

Somebody draws your attention to the fact that the Japanese yield curve is flat and equal to 0%. It also seems that the yen keeps on depreciating against the euro (see Figure 2). So if the euro value of one yen is lower tomorrow than it is today, borrowing money in this currency seems to make sense (the debt will lose value). Let $Z(t)$ be the exchange rate of 1 JPY in EUR, with $Z(0) = 0.006$ EUR/JPY.



Figure 2: EUR/JPY evolution

- (f) Describe a mechanism whereby you would borrow in JPY, change your money into EUR, and redeem the loan in JPY by changing the money back from EUR to JPY in 4 years time. What would be the NPV of this scheme as a function of $Z(4)$?
- (g) For which value of $Z(4)$ is the NPV of that deal positive? What is the NPV of the gain if $Z(4) = Z(0)$? Comment.

3 Equity forward contract

3.1 One date Cash and Carry arbitrage

An equity forward contract exchanges at date T a fixed amount K , agreed upon at date t , against the value S_T of a stock. To hedge a long position in this contract, the idea is to sell the stock short at date t , invest the proceeds in the zero-coupon bond until T , and unwind everything at T . The P&L at date T is

$$(S_T - K) - \left(S_t - \frac{S_t}{P(t, T)} \right) = \frac{S_t}{P(t, T)} - K.$$

This P&L is not random and is obtained for a zero initial investment. If it were different from zero, it would be an arbitrage opportunity. Hence:

$$\begin{aligned} K = F(t, T) &= \frac{S_t}{P(t, T)} \\ &= S_t e^{R(t, T)(T-t)} \\ &\sim S_t e^{r(T-t)} \end{aligned}$$

Definition 8. The value $F(t, T) = \frac{S_t}{P(t, T)}$ is called the **forward value** of the asset. The difference between the forward value and the spot value is called the **basis**.

When interest rates are deterministic, the relative variation of the forward price is equal to the relative variation of the spot price minus the relative variation of the zero-coupon bond:

$$\begin{aligned} dF_t &= d \left(\frac{S_t}{P(t, T)} \right) \\ &= \frac{dS_t}{P(t, T)} - \frac{S_t dP(t, T)}{P(t, T)^2} \\ &= F_t \left(\frac{dS_t}{S_t} - \frac{dP(t, T)}{P(t, T)} \right). \end{aligned}$$

With a constant rate r , $\frac{dP(t, T)}{P(t, T)} = r dt$, so that if the stock has an expected return μ ,

$$\mathbb{E}(dF_t) = F_t (\mu - r) dt :$$

the forward price trends at the risk premium of the stock.

3.2 Intermediate dividends

Consider now a situation where dividends are paid at intermediate dates $(T_1, \dots, T_n)$, proportionally to the forward value at those dates: $\left(\frac{S_t}{P(t, T_1)} d_1, \dots, \frac{S_t}{P(t, T_n)} d_n \right)$. The dividends received while carrying the asset reduce the need for financing, and hence the forward price:

$$\begin{aligned} F(t, T) &= \frac{S_t}{P(t, T)} - \sum_{i=1}^n \frac{S_t d_i}{P(t, T_i)} \frac{P(t, T_i)}{P(t, T)} \\ &= \frac{S_t}{P(t, T)} \left(1 - \sum_{i=1}^n d_i \right). \end{aligned}$$

Each dividend is compounded from its payment date T_i to T , which is the role of the ratio $\frac{P(t, T_i)}{P(t, T)}$.

3.3 Exercises

Exercise 3.1

A share currently trades at 80 USD. Interest rates are equal to 5%, regardless of the maturity. A dividend will be paid in 6 months. Determine the forward price for delivery in 9 months when:

- (a) the dividend equals 0 USD;
- (b) the dividend equals 3 USD;
- (c) the dividend equals 4% of the value of the stock at that time.

Exercise 3.2

The current forward price of a share to be delivered in six months equals 58 USD. Interest rates are equal to 10% per annum, regardless of the maturity. Determine the spot price if:

- (a) no intermediate dividend is paid;
- (b) a dividend of 2 USD is paid in 3 months;
- (c) a dividend equal to 8% of the value of the stock at that time is paid in 3 months;
- (d) a dividend equal to 8% of the value of the stock at that time is paid in 6 months.

Exercise 3.3

The S&P index spot price is 1100 and the 9-month zero-coupon bond price, in % terms, is $ZC = 97.00$. You observe a 9-month forward price of 1109. The dividend is supposed to be paid right before the forward contract maturity.

- (a) What dividend value is implied by this forward price?
- (b) Suppose you believe the dividend that will be paid will be equal to 15. What strategy would you undertake?

4 Futures contracts

4.1 Introduction

For a detailed description of the functioning of futures markets, see chapter 2 of [2]. This section follows [1].

Forward contracts are traded over the counter (OTC), mostly on currencies. On organised exchanges, the dominant contracts are **futures** contracts. Futures are very similar to forwards, but their trading involves a mechanism called **margining**, which changes the nature of their dynamics. As we will see, the difference between forward and futures prices is very much like the difference between buying a zero-coupon bond and rolling over short-term deposits. Let S_t be the price of the underlying at date t .

4.2 Characterisation of the futures price

Let $t_0 = t, t_1, \dots, t_n = T$ be a series of dates, typically days, and denote H_t the futures price. The margining mechanism ensures that when the futures price changes between t_i and t_{i+1} , the party long the futures contract receives $H(t_{i+1}) - H(t_i)$ at date t_{i+1} (and pays it if negative). At maturity, $H(T) = S(T)$.

The futures price differs from the forward price because each variation $H(t_{i+1}) - H(t_i)$ is received at t_{i+1} and has to be reinvested until maturity at a rate which is unknown today. As a result, the futures price cannot be obtained straightforwardly as a function of observable quantities, as is the case for forward prices.

To better understand the nature of futures prices, we will describe a strategy whose initial value is the futures price and whose final value can be explicitly described. To do so, we first identify such a strategy for forward contracts, and then extend it to futures contracts.

4.2.1 Characterisation of forward prices

At date t , the forward price of maturity T can be interpreted as the spot price of $1/P(t, T)$ units of stock. Put differently:

Proposition 2. *The forward price is the price at date t of a payoff paying $\frac{S(T)}{P(t, T)}$ at date T .*

Another way to obtain this result is to consider the following strategy:

1. at date t , go long $\frac{1}{P(t, T)}$ forward contracts and hold them until maturity;
2. invest an amount equal to the forward price $F(t, T)$ in the zero-coupon bond of maturity T .

The initial cost of this strategy is $F(t, T)$, since forward contracts cost nothing, and its value at date T is

$$\begin{aligned} \frac{F(t, T)}{P(t, T)} + \frac{1}{P(t, T)}(F(T, T) - F(t, T)) &= \frac{F(T, T)}{P(t, T)} \\ &= \frac{S(T)}{P(t, T)}. \end{aligned}$$

The proposition then follows from the law of one price.

4.2.2 Extension to futures prices

In the case of futures, the idea is to repeat the previous strategy over each period $[t_i, t_{i+1}]$, until T . Let us write $P(i, i+1)$ for $P(t_i, t_{i+1})$ and $H(i)$ for $H(t_i)$. Starting at t_0 :

1. At t_0 , go long $\frac{1}{P(0,1)}$ futures contracts and invest $H(0)$ in the riskless zero-coupon bond maturing at t_1 .
2. At t_1 , unwind the strategy. The corresponding wealth is

$$\frac{1}{P(0,1)}(H(1) - H(0)) + \frac{H(0)}{P(0,1)} = \frac{H(1)}{P(0,1)}.$$

At t_1 , repeat the same strategy, now with $\frac{H(1)}{P(0,1)}$ (and not $H(0)$) as initial wealth:

1. At t_1 , go long $\frac{1}{P(0,1)P(1,2)}$ futures contracts and invest $H(1)/P(0,1)$ in the riskless zero-coupon bond maturing at t_2 .
2. At t_2 , unwind the strategy. The corresponding wealth is

$$\frac{1}{P(0,1)P(1,2)}(H(2) - H(1)) + \frac{H(1)}{P(0,1)P(1,2)} = \frac{H(2)}{P(0,1)P(1,2)}.$$

Let us now introduce the rolled-over discount factor between t_i and t_j : $B(t_i, t_j) := \prod_{k=i}^{j-1} P(k, k+1)$, so that $1/B(t_i, t_j)$ is the value at t_j of 1 invested at t_i and rolled over in the short-term riskless account. Note that $B(t_i, t_j)$ is random seen from t_i , since the future short-term rates are not known yet. At each t_i , the strategy consists in:

1. at date t_i , go long $\frac{1}{B(t_0, t_{i+1})}$ futures contracts and invest $\frac{H(i)}{B(t_0, t_i)}$ in the riskless zero-coupon bond maturing at t_{i+1} ;
2. at date t_{i+1} , unwind the strategy. The corresponding wealth is

$$\frac{1}{B(t_0, t_{i+1})}(H(i+1) - H(i)) + \frac{H(i)}{B(t_0, t_i) \times P(i, i+1)} = \frac{H(i+1)}{B(t_0, t_{i+1})}.$$

Iterating until $t_n = T$, where $H(T) = S(T)$, the final value of this strategy is

$$\frac{S(T)}{B(t_0, T)}.$$

Its initial cost is $H(0)$. Hence, by the law of one price:

Proposition 3. *The futures price is the price at date t of a payoff paying $\frac{S(T)}{B(t, T)}$ at date T .*

Futures and forward prices look very much alike: both can be understood as the spot price amplified by the cost of carrying the asset. However, for the forward price, the cost of carry is known from the start, while it is random for the futures price.

Proposition 4. *When interest rates are deterministic, $B(t, T) = P(t, T)$ and futures and forward prices are equal.*

A further study of the difference between futures and forward prices requires a more detailed modelling of interest rates.

4.3 Exercises

Exercise 4.1 *Problem: benefiting from the Abenomics*

In 2012 and 2013, the Japanese government led by Shinzo Abe adopted massive stimulus plans, sometimes referred to as Abenomics.

As a result, the Nikkei 225 appreciated by approx. +57% in 2013. On the contrary, the yen depreciated by approx. -22% against the USD, moving from 87 JPY/USD to 105 JPY/USD. In 2013, interest rates were approximately equal to zero both in JPY and in USD.

How could a US investor benefit from the Abenomics? Consider the two following strategies:

- Strategy 1:
 - Jan 1st, 2013: borrow USD 1M, change it spot into JPY and buy a Nikkei 225 ETF;
 - Dec 31st, 2013: sell the Nikkei ETF, change the proceeds spot into USD and repay the USD loan.
- Strategy 2:
 - Jan 1st, 2013: borrow JPY 87M and buy a Nikkei 225 ETF;
 - Dec 31st, 2013: sell the Nikkei ETF, repay the JPY loan and change the proceeds spot into USD.

(a) Compute the final USD value of both strategies. Discuss.

(b) Consider the two following strategies:

- Strategy 3:
 - Jan 1st, 2013: “buy” a 1Y futures contract on the Nikkei 225 with notional JPY 87M;
 - Dec 31st, 2013: receive the futures payment at expiry and change it spot into USD.
- Strategy 4:
 - Jan 1st, 2013: “buy” a 1Y CME futures contract on USD/JPY with notional USD 1M;
 - Dec 31st, 2013: receive the futures payment at expiry.

Compute the P&Ls of strategies 3 and 4. Can you combine them?

(c) Eventually, what would have been an appropriate Abenomics “trade”?

5 Pricing models, Risk Neutral Probability

5.1 Risk neutral probability

In corporate finance, the value of a project is usually computed as the “expected value of its cash flows”, discounted at a rate “corresponding to their class of risk”. But under which probability should this expectation be taken?

The existence of forward markets provides another way of looking at this question. Consider a probability measure under which one would like to compute the “expected value of cash flows”. This probability measure has to give a zero price to any forward contract or, more generally, to any contingent claim with zero NPV. In other words, it has to be consistent with the absence of arbitrage.

Proposition 5 (No Arbitrage). *The following propositions should hold:*

1. *The price of any cash flow with zero NPV has to be equal to zero;*
2. *The expected value of an asset under the “pricing measure” should be its forward value.*

Let us look for a price functional giving the price of a random flow $\tilde{X}_T$ paid at date T , and see which constraints it has to satisfy. In the absence of arbitrage, the price of a portfolio is the sum of the prices of its components, so that this functional is linear. We look for it in the form

$$\text{Price}(\tilde{X}_T) = kE^Q(\tilde{X}_T)$$

where k is a real number and Q a measure, not necessarily positive a priori.

First, consider the claim which pays 1 if the event A occurs and zero otherwise, denoted 1_A . The price of this claim has to be strictly positive, otherwise it would be a free lunch. As a result

$$\forall A : kE^Q(1_A) > 0$$

which means that Q is a positive measure. We scale Q so that it is a probability, i.e. $E^Q(1) = 1$.

Second, when $A = \Omega$, $1_\Omega = 1$ almost surely and the claim is the zero-coupon bond:

$$\text{Price}(1) = k = P(0, T).$$

Finally, let $\tilde{X}(T)$ be the price of an asset at date T and $F(T)$ its forward price. By definition of the forward price, entering a forward contract costs nothing:

$$\begin{aligned} \text{Price}(\tilde{X}(T) - F(T)) &= 0 \\ \Leftrightarrow E^Q(\tilde{X}_T) &= F(T). \end{aligned}$$

This can be summarised in the following proposition:

Proposition 6 (Fundamental Pricing Theorem). *In the absence of arbitrage, any claim can be priced as the discounted expected value of its cash flows under a probability Q under which the expected value of any asset is equal to its forward price. In particular, for any asset with price $\tilde{S}(T)$ whose forward price is $F(T)$, and for any function $g: \mathbb{R} \rightarrow \mathbb{R}$, one has*

$$\text{Price}(g(\tilde{S}(T))) = P(0, T) E^Q(g(\tilde{S}(T)))$$

and

$$E^Q(\tilde{S}(T)) = F(T)$$

Q is called the **risk neutral probability** (RNP): under Q , all assets have the same expected return, equal to the riskless rate.

5.2 Examples

5.2.1 Stocks

Let $\tilde{S}(T)$ be the value of a stock which does not pay dividends. One has

$$E^Q(\tilde{S}(T)) = \frac{S(0)}{P(0, T)} = S_0 e^{rT}$$

Cash and carry with a dividend and a carrying cost. Before turning to dividends paid continuously, let us come back to the cash and carry argument of Section 3.1 in a slightly more realistic setting. Holding one unit of the underlying between 0 and T now generates two intermediate flows, both known at date 0:

- a dividend D received at date T_D ;
- a carrying cost C paid at date T_C (storage and insurance for a commodity, custody fees for a security, etc.).

with $0 < T_D, T_C \leq T$ (their relative order does not matter). Consider a forward contract of maturity T and strike K , and let us **sell** it. The following strategy hedges this sale operation by operation. To ease notation, let

$$V_0 \equiv S_0 - D P(0, T_D) + C P(0, T_C)$$

be the amount which has to be financed at date 0 to carry the underlying until T .

Operation	0	T_D	T_C	T
Sell the forward at K	0			$K - S_T$
Buy the underlying	$-S_0$	$+D$	$-C$	$+S_T$
Borrow until T_D the present value of the dividend	$+D P(0, T_D)$	$-D$		
Lend until T_C the present value of the cost	$-C P(0, T_C)$		$+C$	
Borrow the residual amount until T	$+V_0$			$-\frac{V_0}{P(0, T)}$
Total	0	0	0	$K - \frac{V_0}{P(0, T)}$

Table 3: Cash and carry arbitrage with a dividend D paid at T_D and a carrying cost C paid at T_C : cash flows of each operation at each date.

Each line of Table 3 has a simple interpretation:

- Buying the underlying is the hedge of the forward sale: at T , the unit of underlying held is exactly what has to be delivered, so that S_T disappears from the total. The forward itself costs nothing at inception.
- The dividend is “consumed” in advance: one borrows its present value today and repays the loan with the dividend at T_D . The dividend thus **reduces** the amount to be financed.
- Symmetrically, the money needed to pay the cost at T_C is set aside today in a deposit. The cost thus **increases** the amount to be financed.
- Whatever is still needed, V_0 , is borrowed until T .

The strategy is self-financing: the intermediate columns all sum to zero, and the only net flow, at date T , is $K - V_0/P(0, T)$, which is known as of date 0. By the absence of arbitrage, it has to be zero and the forward price is

$$F(0, T) = \frac{V_0}{P(0, T)} = \frac{S_0}{P(0, T)} - D \frac{P(0, T_D)}{P(0, T)} + C \frac{P(0, T_C)}{P(0, T)},$$

that is the spot price compounded until T , minus the dividend compounded from T_D to T , plus the cost compounded from T_C to T .

What if the forward is quoted at another price?

- If $K > F(0, T)$, the forward is too expensive. One sells it and runs the strategy of Table 3 (**cash and carry**): for a zero initial investment, one gets the riskless profit $K - F(0, T) > 0$ at date T . Arbitrageurs selling the forward and buying the underlying push K down and S_0 up until the opportunity disappears.
- If $K < F(0, T)$, the forward is too cheap. One buys it and runs every line of the table in reverse (**reverse cash and carry**): sell the underlying, lend the proceeds until T , etc. The profit at T is $F(0, T) - K > 0$. Note that a short seller borrows the security and has to pay the dividend D to the lender at T_D (“manufactured dividend”), exactly as in the reversed table. The saving of the carrying cost C , however, only accrues to someone who actually holds the underlying and sells it.

The forward price is thus pinned down from both sides. In practice, the two strategies do not have the same cost: bid-ask spreads, borrowing rates higher than lending rates, fees to borrow the security, etc. They only define a **no-arbitrage band** around $F(0, T)$ inside which the forward price can fluctuate. The asymmetry is particularly strong for commodities: the reverse cash and carry requires to own inventories, which is why commodity forwards may durably trade below their cash and carry value (this gap is known as the **convenience yield**).

Consequence for the risk neutral probability. By Proposition 6,

$$E^Q \left(\tilde{S}(T) \right) = F(0, T) = \frac{S_0 - D P(0, T_D) + C P(0, T_C)}{P(0, T)}.$$

When $D = C = 0$ one recovers the previous formula. A dividend lowers the risk neutral expectation of the future price: since holding the stock gives both the dividend and the final value, the final value alone has to grow at a rate lower than r . Conversely, a carrying cost raises it.

Let us now consider the limit case where dividends are paid continuously, proportionally to the value of the stock, at a rate d (continuous dividend yield). What should then be the risk neutral trend μ of the stock? Buying one unit of stock at date 0 gives right to:

1. the value of the stock at any future date t ;
2. the flow of dividends $d S_s ds$ received between 0 and t .

As a result, under the RNP, with $\mathbb{E}^Q(S_s) = S_0 e^{\mu s}$:

$$\begin{aligned} S_0 &= E \left(\int_0^t d e^{-rs} S_s ds \right) + E(e^{-rt} S_t) \\ &= S_0 \times \left(d \int_0^t e^{(\mu-r)s} ds + e^{(\mu-r)t} \right) \\ &= S_0 \times \left(\frac{d}{\mu-r} (e^{(\mu-r)t} - 1) + e^{(\mu-r)t} \right) \end{aligned}$$

The only way for this equality to hold for any t is

$$\mu = r - d.$$

So under the RNP, a dividend paying asset trends at the rate $r - d$:

$$E^Q(S_t) = S_0 e^{(r-d)t}.$$

5.2.2 Currencies

For an exchange rate $X(T)$ giving the value in foreign currency of 1 unit of domestic currency, the expected value is the forward exchange rate:

$$E^Q \left(\tilde{X}(T) \right) = X(0) \frac{P^{domestic}(0, T)}{P^{foreign}(0, T)} = X_0 e^{(r^f - r^d)T}.$$

5.3 The scaling of variance

We would like to model the price of an asset as the exponential of a Gaussian random variable:

$$\tilde{S}_T \stackrel{\text{law}}{=} e^{Y(T)}$$

where $Y(T) \sim N(\mu(T), \sigma^2(T))$. What is a natural choice for the function $\sigma^2(T)$? We choose an arbitrary subdivision of $[0, T]$ into n sub-intervals of length $\delta = T/n$: $\tau_n = \{0, \delta, 2\delta, \dots, n\delta = T\}$, and write

$$Y(T) = Y(0) + \sum_{i=1}^n \Delta Y_i, \quad \Delta Y_i \equiv Y(i\delta) - Y((i-1)\delta).$$

We make two assumptions on the increments ΔY_i :

1. they are drawn from the same law (stationarity), with finite variance;
2. they are independent.

Stationarity means that the variance of an increment only depends on the length of the interval. Independence then yields that the variances add up:

$$\sigma^2(t_1 + t_2) = \sigma^2(t_1) + \sigma^2(t_2)$$

so that $\sigma^2(T)$ is linear in T . Denoting $\sigma^2 \equiv \sigma^2(1)$ the variance per unit of time:

$$\begin{aligned} \sigma^2(T) &= \sigma^2 T \\ \sigma(T) &= \sigma \sqrt{T} \end{aligned}$$

The variance grows linearly with the horizon, and the volatility as its square root.

5.4 The Black and Scholes model

Theorem 1 (Gaussian Laplace Transform). Let $X \sim N(\mu, \sigma^2)$. For any $\lambda \in \mathbb{R}$,

$$E(e^{\lambda X}) = e^{\lambda\mu + \frac{\lambda^2\sigma^2}{2}}.$$

By Theorem 1, choosing for a stock which does not pay dividends

$$\begin{aligned} Y(T) &\sim N\left(rT - \frac{1}{2}\sigma^2T, \sigma^2T\right) \\ \Rightarrow E\left(S_0 e^{Y(T)}\right) &= S_0 e^{rT} \end{aligned}$$

ensures that the expected value of the stock is its forward price, as required under the RNP. The correction $-\frac{1}{2}\sigma^2T$ of the mean compensates for the convexity of the exponential. All the above can also be obtained through stochastic calculus.

The Black and Scholes model is a model where the underlying price follows a log-normal diffusion, with an expected return μ under the historical probability P and r under the risk neutral probability Q :

$$\begin{aligned} dS_t &= S_t (\mu dt + \sigma dW_t^P) \\ &= S_t (r dt + \sigma dW_t^Q) \end{aligned}$$

and where interest rates are constant, equal to r . In this model,

$$\begin{aligned} S_t &= S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t^P} \\ E^P(S_t) &= S_0 e^{\mu t} \\ E^Q(S_t) &= S_0 e^{rt} \end{aligned}$$

Time-varying and stochastic volatility. Nothing in the above requires the volatility to be constant. If, under Q ,

$$dS_t = S_t (r dt + \sigma_t dW_t),$$

then

$$S_T = S_0 \exp\left(rT - \frac{1}{2} \int_0^T \sigma_s^2 ds + \int_0^T \sigma_s dW_s\right).$$

When σ_t is deterministic, $\int_0^T \sigma_s dW_s \sim N\left(0, \int_0^T \sigma_s^2 ds\right)$: everything works as before, with σ^2T replaced by $\int_0^T \sigma_s^2 ds$. When σ_t is stochastic but independent of W , the same holds conditionally on the path of the volatility, and still $E^Q(S_T) = S_0 e^{rT}$. In both cases, σ^2T should be interpreted as the **integrated variance** along the path, and σ^2 as the **average variance** over $[0, T]$:

$$\sigma^2 = \frac{1}{T} \int_0^T \sigma_s^2 ds.$$

6 Option pricing as discounting expected cash flows

6.1 Definitions and properties

As opposed to futures and forwards, options are contracts which the holder may or may not exercise: they give a right, not an obligation.

Definition 9. The right to buy a stock at some future date T at some predefined price K is called a **call option**. The pre-agreed price K is called the **strike price** and T the **maturity** of the option. The right to sell a stock at a predefined strike price is called a **put option**.

Definition 10. A **European option** is a payoff which is paid at a given date T , with no intermediate payment or exercise. An **American option** can be exercised at any moment between its inception and its maturity.

One should keep in mind that, as shown in Proposition 6, the historical probability P is not the probability under which pricing is made. Pricing is necessarily made under a probability which ensures that the expected future value of any underlying is equal to its forward value, i.e. to its spot value increased by the cost of carry.

Definition 11. A **payoff** is a function g which describes the amount to be paid as a function of one or several observable variables: the final value of the asset, its maximum or minimum, its average, the event of having hit a barrier, etc. The payoffs of call and put options are

$$\begin{aligned} \text{Call}(S_T, K) &= (S_T - K)^+ \\ \text{Put}(S_T, K) &= (K - S_T)^+ \\ \text{with } x^+ &= \max(x, 0). \end{aligned}$$

Proposition 7 (Call put parity). Since

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K,$$

a long call and a short put with the same strike and maturity are equivalent to a long forward contract of strike K :

$$\text{call}(S_0, K, T) - \text{put}(S_0, K, T) = P(0, T)(F(0, T) - K).$$

When no dividends are paid, $P(0, T)F(0, T) = S_0$ and

$$\begin{aligned} \text{call}(S_0, K, T) - \text{put}(S_0, K, T) &= S_0 - KP(0, T) \\ &= S_0 - Ke^{-rT}. \end{aligned}$$

6.2 Factors affecting option prices

We consider European payoffs g which are functions of the final value S_T of an asset paying a continuous dividend. The price of such a payoff is

$$\pi(S_0) = e^{-rT} E^Q(g(S_T)).$$

Definition 12 (Intrinsic value). The **intrinsic value** of a payoff is $IV = e^{-rT} g(E^Q(S_T)) = e^{-rT} g(F(0, T))$. It is the price the payoff would have in a model with no randomness.

Definition 13 (Time value). The **time value** of a payoff is the difference between its price and its intrinsic value:

$$\text{Price} = IV + TV.$$

The sign of the time value is related to the convexity of g (Jensen's inequality). When g is convex, the time value is positive. When it is concave, the time value is negative.

For calls and puts, the time value:

- is maximal at the money (K close to the forward);
- tends to 0 when the option is deep in or deep out of the money: the option then behaves either as a forward contract or as nothing;
- increases with the volatility and, generally, with the maturity, since both increase the uncertainty on the final value of the asset.

6.3 Example of strategies involving options

Several payoff examples can be found in this document.

6.4 B&S formulas

Let N denote the cumulative distribution function of the standard Gaussian law.

The B&S formula without dividends

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2), \quad (3)$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1), \quad (4)$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad (5)$$

$$d_2 = d_1 - \sigma\sqrt{T}. \quad (6)$$

The B&S formula with dividends With a continuous dividend yield q (denoted d in Section 5):

$$C = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2), \quad (7)$$

$$P = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1), \quad (8)$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad (9)$$

$$d_2 = d_1 - \sigma\sqrt{T}. \quad (10)$$

The B&S formula with forward and zero-coupon as inputs (aka the Black formula)

$$C = P(0, T) (FN(d_1) - KN(d_2)), \quad (11)$$

$$P = P(0, T) (KN(-d_2) - FN(-d_1)), \quad (12)$$

with

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}, \quad (13)$$

$$d_2 = d_1 - \sigma\sqrt{T}. \quad (14)$$

The two previous formulas are special cases of the Black formula, with $P(0, T) = e^{-rT}$ and $F = S_0 e^{rT}$ or $F = S_0 e^{(r-q)T}$.

6.5 Options on different underlyings

The Black formula only requires the forward price of the underlying and the zero-coupon bond of the option maturity. It therefore applies to any underlying, provided the relevant forward is used:

- stock without dividends: $F = S_0/P(0, T)$;
- stock with a continuous dividend yield q : $F = S_0 e^{(r-q)T}$;
- stock with discrete dividends or asset with carrying costs: F given by the cash and carry argument (Table 3);
- currency: $F = X_0 \frac{P^{domestic}(0, T)}{P^{foreign}(0, T)}$;
- futures contract: $F = H$, the futures price, when interest rates are deterministic.

σ is then the volatility of the forward price.

6.6 Exercises

Exercise 6.1

(Use a B&S calculator available on the Web.) Determine the price of a European futures call option on a barrel of crude oil to be delivered in 4 months, which is also the maturity of the option. The futures price today is $F_0 = \$25.00$, the strike price is $\$23.00$ and the volatility of the futures price is estimated to be 25% per year. The risk-free interest rate is 9% per year.

Exercise 6.2

(Use a B&S calculator available on the Web.)

- Calculate the price of a European call option on a share which does not pay dividends before the maturity of the option. The spot share price is 45 SEK, the option matures in 6 months, the volatility is 25% per annum, the strike price is 43 SEK and the risk-free interest rate is 5% per annum.
- Same question, assuming now that the share pays a dividend of 1 SEK in 3 months, all other assumptions being unchanged.

Exercise 6.3

Denote X_t the USD vs EUR exchange rate at date t , i.e. the value of 1 USD in EUR: 1 USD = X_t EUR. We denote $r_{\text{€}}$ and $r_{\text{§}}$ respectively the EUR and the USD continuously compounded rates. In what follows, $F(t, T)$ denotes the forward value of X_t for date $T \geq t$. EUR is the domestic currency in which all prices are denominated.

- (a) Give the definition of the forward price $F(t, T)$. At which date is it known? At which date is it paid?
- (b) Show that $F(t, T) = X_t e^{(r_{\text{€}} - r_{\text{§}})(T-t)}$.

We now wish to price a call with strike K and maturity T on X_T . Instead of recalculating everything, we would like to adapt the Black and Scholes formula.

- (c) Use the previous formula to find the price of a call option on the exchange rate as a function of $r_{\text{€}}$, $r_{\text{§}}$, X_0 , σ and T .
- (d) Compare the intrinsic value of the call and the amount of money you could get by exercising it. Discuss when it makes sense to exercise it or not.
- (e) Same question for the put.

Exercise 6.4

You are given the following:

- the current price of one share of XYZ stock is 500;
- the stock does not pay dividends;
- interest rates are null;
- a European call option on one share of XYZ stock with a strike price of K that expires in one year costs 67;
- a European put option on one share of XYZ stock with a strike price of K that expires in one year costs 19.

Find K .

Exercise 6.5

The PS index has the following characteristics:

- one share of the PS index currently sells for 1,000;
- the PS index does not pay dividends.

Sam wants to secure the ability to buy this index in one year for a price of 1,025 (even if the price eventually gets lower than 1,025). He can do so by buying or selling European put and call options with a strike price of 1,025. The annual effective risk-free interest rate is 5%. Determine which of the following gives the hedging strategy that will achieve Sam's objective, and calculate the cost today of establishing this position.

1. Buy the put and sell the call, receive 23.81
2. Buy the call and sell the put at no cost
3. Buy the call and sell the put, receive 23.81
4. Buy the call and sell the put, pay 23.81

Instead of 1,025, which strike would allow to secure the forward purchase at zero cost?

Exercise 6.6

Near market closing time on a given day, you lose access to stock prices, but some European call and put prices for a stock are available as follows:

Strike price	Call price	Put price
40	11	3
50	6	8
55	3	11

Explain why this pricing table would provide some arbitrage. Propose an arbitrage strategy. (Hint: use the call-put parity.)

Exercise 6.7

Let X_t be the value of a foreign currency expressed in units of local currency. Let r_f be the (constant) interest rate of the foreign currency and r_l the local interest rate. Let $T \geq t$ be a maturity.

- (a) Compute the forward value of X_t .
 - (a) Does it increase or decrease with r_f ? With r_l ?
 - (b) What is the “cost of carry” of such an “asset”?
 - (c) What would be the risk neutral drift of that exchange rate?
- (b) Write the equivalent of the Black and Scholes formula in such a situation (solution: https://en.wikipedia.org/wiki/Foreign_exchange_option).

Exercise 6.8

- (a) In a world without dividends, prove the call-put parity, i.e.

$$KD(0, T) + \text{Call}(S_0, K, T, r) = S_0 + \text{Put}(S_0, K, T, r)$$

where $D(0, T) = e^{-rT}$ is the discount factor, S_0 is the initial value of the price, and $\text{Call}(S_0, K, T, r)$ (resp. $\text{Put}(S_0, K, T, r)$) is the price of the call (resp. put) of strike K and maturity T obtained with the B&S formula with interest rate r .

- (b) Let F be the forward price of S : $F(t, T) = S_t e^{r(T-t)}$. Thanks to Itô's lemma, compute the drift of $F(t, T)$ under the risk neutral probability. Denote $F = F(0, T)$.
- (c) Prove that, equivalently,

$$K + \text{Call}(F, K, T, 0) = F + \text{Put}(F, K, T, 0)$$

where the options are now options on the futures contract.

Exercise 6.9 Problem: the Merton model of the firm

This model aims at providing the value of the debt of a company, taking into account its bankruptcy risk. The company is modelled as follows. Its business value at time t is denoted S_t , and its initial business value S_0 . The volatility of the business is denoted σ . The company has been financed by two means:

1. some initial capital, E_0 , from shareholders;
2. some initial debt in the form of a zero-coupon bond of notional D and maturity T , from creditors.

The company will not pay any intermediate dividend. The riskless interest rate is denoted r . The interest rate paid by the company equals $r + s$, where s is a spread above the riskless rate which reflects the fact that lending money to the company is risky. This means that the value of the risky zero-coupon bond today is $De^{-(r+s)T}$. In this framework, the bankruptcy event is defined as $S_T < D$.

At maturity, shareholders and creditors share the business value S_T as follows:

1. Creditors are paid first. If the company is not bankrupt, they are repaid their notional D . If the company is bankrupt, they take the whole company value.
2. Shareholders are paid second: if the company is not bankrupt, they get $S_T - D$; if it is bankrupt, they receive 0.

Let F_T be the T -forward value of the business. The debt-to-asset ratio, defined as $l = \frac{D}{F_T}$, is used as a measure of leverage. When $l = 0$, the company is fully unlevered; when it equals 1, it is infinitely leveraged.

- Write the final wealth of shareholders as an option payoff. Give the name of this option. Using the Black and Scholes formula, express the value of equity as a function of the interest rate r , the leverage l , the volatility σ and the maturity T .
- Using the fundamental law of accounting, express the value of the risky debt. Then, using the call-put parity, find an expression of the spread.
- Compute the value of the spread s and of the equity for the following sets of parameters, and comment:

σ	l	T	r
20%	1/2	1	0
20%	3/4	1	0
40%	1/2	1	0

- More generally, who has an interest in the company taking more risk ($\sigma \nearrow$)?
- We assume $T = 1$, $l = 3/4$, $\sigma = 30\%$ and $r = 0\%$. At date 0, the company announces that it will pay to shareholders an intermediate dividend equal to 5% of the initial value S_0 . What is the new forward value of the company? What is the new leverage? Compute the new value of the share price and of the spread. Comment.

Exercise 6.10 Problem: hedging a corporate FX exposure

Background: a multinational corporation (MNC) based in France expects to receive USD 100 million from a U.S. client in 3 months. The CFO is concerned about exchange rate fluctuations, especially after recent unexpected macroeconomic news which led to increased volatility in the USD/EUR market. By USD/EUR, we mean the value of 1 USD in EUR, denoted X , with $X_0 = 0.9$ today. The future (random) value of the USD/EUR rate in 3 months is denoted $\tilde{X}_3$.

According to the CEO, the USD should appreciate against the EUR over the next 3 months, due to expectations of a tighter monetary policy by the FED. On the contrary, the CFO thinks there will be less demand for USD as a safe-haven currency, and hence that the USD should depreciate. They are not very sure about what to do...

US rates are equal to $r_{USD} = 5\%$ and the euro rate is $r_{EUR} = 1\%$. The annual volatility of the USD/EUR is $\sigma = 10\%$. Rates are continuously compounded.

- Does X increase or decrease when the USD appreciates? If the CEO is right, should you hedge the risk or not? Same question if the CFO is right.
- What is the forward value of the USD/EUR rate? Write the formula and calculate a numerical approximation. In EUR, what difference would it make to change USD 100 million at the spot rate or at the forward rate? If the future value of the USD/EUR equals the forward rate, will it mean that the USD has appreciated or depreciated?
- Explain why the volatility of $\tilde{X}_3$ should be equal to 5%. What is the typical fluctuation of the EUR equivalent of USD 100 million in three months?
- If you do not want to take any risk, what should you do? Will you obtain the support of the CEO or of the CFO?
- MNC hesitates between buying a put option or a call option on the USD/EUR. Which one would be appropriate? Explain why.
- Eventually, the company purchases a European put option on USD 50 million with a strike price of 0.9 USD/EUR, expiring in 3 months. The price of that option equals 1.5% of the notional amount. Graph the following values as a function of X_3 :
 - the EUR value of the USD 100 million in three months;
 - the EUR value of the put portfolio, including its premium;
 - the sum of both.
- Compare this strategy with a full hedge (changing the USD forward). Comment. What would be your own view on the matter?

7 Option pricing from hedging: the Black & Scholes approach (complex)

7.1 A PDE for option prices

Seminal papers: [4] and [3].

The asset is assumed not to pay any dividend. Let

- S be the price process, with expected return μ and volatility σ ;
- K be the strike price of the option;
- r be the constant positive interest rate;
- T be the maturity of the option;
- W be a Brownian motion.

C or $C(t, S_t)$ denotes the call price and P or $P(t, S_t)$ the put price. When necessary, C^a and C^e distinguish between American and European exercise. At maturity, $C(T, x) = (x - K)^+$ and $P(T, x) = (K - x)^+$.

The spot price is the solution of the following stochastic differential equation, with initial value S_0 :

$$dS_t = S_t(\mu dt + \sigma dW_t). \quad (15)$$

Locally, the variations of the option price are given by Itô's formula:

$$dC = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right] dt + \frac{\partial C}{\partial S} \sigma S dW_t.$$

Now consider a portfolio made of one option and δ stocks. Its value V varies as follows:

$$dV = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \delta \mu S \right] dt + \sigma S \left(\frac{\partial C}{\partial S} + \delta \right) dW_t.$$

If, at each date t , $\delta = -\frac{\partial C}{\partial S}$, the Brownian terms disappear and the portfolio bears no risk between t and $t + dt$. Its instantaneous return must then be equal to $r dt$, since any other value would create an arbitrage opportunity: $dV = rV dt$ with $V = C - \frac{\partial C}{\partial S} S$. The price of the option therefore satisfies a partial differential equation, which no longer depends on μ :

$$\begin{aligned} \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} &= rC \\ C(T, x) &= (x - K)^+ \end{aligned} \quad (16)$$

Equation (16) is known as the **Black and Scholes equation**. How can it be solved?

7.2 Solving the PDE

The PDE can be solved by standard methods, such as Fourier transforms, or numerically.

The solution of a linear PDE can also be interpreted as the expectation of some function under some probability, which connects this approach to the previous one. This relationship between PDEs and expectations is known as the Feynman-Kac approach. Let us sketch how it works.

7.2.1 The Feynman-Kac equivalence

Define a process

$$\begin{aligned} dX(t) &= \mu(t, X) dt + \sigma(t, X) dW_t \\ X(0) &= X_0 \end{aligned} \quad (17)$$

with μ and σ satisfying a Lipschitz condition, so that equation (17) has a unique strong solution, and its infinitesimal generator

$$\mathcal{L}f(t, x) = \frac{\partial f}{\partial t} + \mu(t, x) \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 f}{\partial x^2}. \quad (18)$$

For any $C^{1,2}$ function f , Itô's formula yields

$$df(t, X) = \mathcal{L}f dt + \frac{\partial f}{\partial x} \sigma(t, X) dW_t.$$

Now define

$$Z_t = f(t, X_t) \equiv \mathbb{E}(F(X_T) | \mathcal{F}_t). \quad (19)$$

By the tower property, for any $u > t$,

$$\mathbb{E}(F(X_T) | \mathcal{F}_t) = \mathbb{E}(\mathbb{E}(F(X_T) | \mathcal{F}_u) | \mathcal{F}_t)$$

or equivalently

$$\begin{aligned} f(t, X_t) &= \mathbb{E}(f(u, X_u) | \mathcal{F}_t) \\ Z_t &= \mathbb{E}(Z_u | \mathcal{F}_t) : \end{aligned}$$

Z is a martingale. Define the Cauchy problem (a PDE with a terminal condition):

$$P(\mathcal{L}, F) = \begin{cases} \mathcal{L}f & = 0 \\ f(T, x) & = F(x) \end{cases}$$

For $f(t, X_t)$ to be a martingale with final value $F(X_T)$, it must not have any finite variation part, hence $\mathcal{L}f(t, X_t) = 0$ for any possible value of X_t , i.e. $\mathcal{L}f = 0$. As a result, f is a solution of the Cauchy problem $P(\mathcal{L}, F)$.

Conversely, if f is a solution of the Cauchy problem $P(\mathcal{L}, F)$, then $f(t, X_t)$ is a martingale with final value $F(X_T)$, i.e. f is given by equation (19).

7.2.2 Application to the B&S PDE

Equation (16) does not immediately define such a Cauchy problem, because of the term rC on the right-hand side.

To remove this term, we introduce the forward price of the option, $\tilde{C}(t, x) \equiv e^{r(T-t)}C(t, x)$. The space derivatives of $\tilde{C}$ and C are proportional: $\frac{\partial \tilde{C}}{\partial x} = e^{r(T-t)} \frac{\partial C}{\partial x}$. The time derivatives, however, differ because of the discounting: $\frac{\partial \tilde{C}}{\partial t} = e^{r(T-t)} \times \left(\frac{\partial C}{\partial t} - rC \right)$.

Multiplying equation (16) by $e^{r(T-t)}$ therefore yields an equation on the forward price of the option:

$$P(\mathcal{L}^{BS}, (x - K)^+) = \begin{cases} \mathcal{L}^{BS} \tilde{C} & = 0 \\ \tilde{C}(T, x) & = (x - K)^+ \end{cases}$$

where $\mathcal{L}^{BS}$ is defined as in equation (18) with $\mu(t, x) = rx$ and $\sigma(t, x) = \sigma x$. By the Feynman-Kac equivalence, $\tilde{C}(t, x) = \mathbb{E}\left((S_T - K)^+ | S_t = x\right)$ where $dS_t = S_t(rdt + \sigma dW_t)$, that is

$$C(t, x) = e^{-r(T-t)} \mathbb{E}^Q\left((S_T - K)^+ | S_t = x\right) :$$

we recover the risk neutral pricing formula.

With three further changes of variables:

- forward underlying price instead of spot underlying price: $x \rightarrow e^{r(T-t)}x$;
- log forward price instead of forward price: $x \rightarrow \ln(x)$;
- removal of the drift $-\frac{1}{2}\sigma^2$ of the log forward price: $y \rightarrow y - \frac{1}{2}\sigma^2(T-t)$;

equation (16) becomes, for the forward price of the option $u(t, y)$:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2} = 0 \quad (20)$$

$$u(T, y) = (e^y - K)^+ \quad (21)$$

which is known as the **heat equation**.

8 Hedging and the greeks

8.1 Definition and representation

Definition. The **greeks** are the sensitivities of the price of an option to its parameters: $\delta = \frac{\partial C}{\partial S}$ (delta), $\Gamma = \frac{\partial^2 C}{\partial S^2}$ (gamma), $\mathcal{V} = \frac{\partial C}{\partial \sigma}$ (vega), $\Theta = \frac{\partial C}{\partial t}$ (theta) and $\rho = \frac{\partial C}{\partial r}$ (rho). They are summarised in Table 4.

Graphical representation. Figure 3 represents the price of a call as a function of the spot. At the current spot S , the tangent to the price curve has slope δ and intercept $C - \delta S$. A portfolio made of δ stocks and of an amount $C - \delta S$ in cash (negative for a call: it is a borrowing) has the same value as the option and the same sensitivity to small moves of the spot: it is the **hedging** (or replicating) **portfolio** of the option.

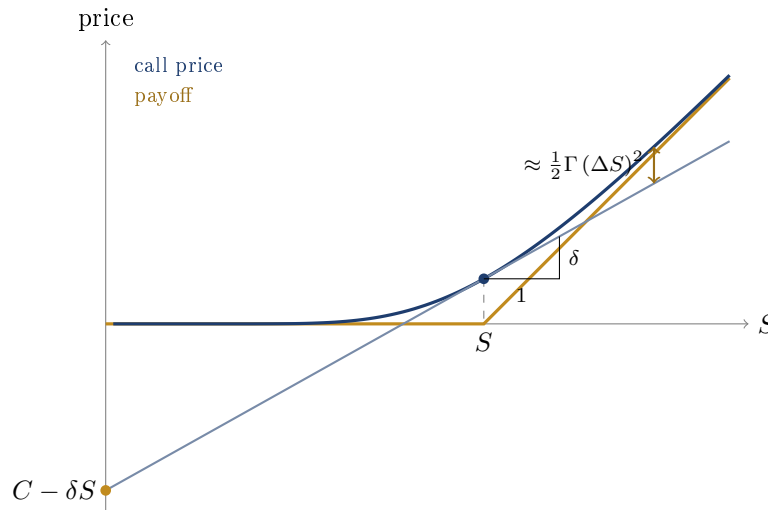


Figure 3: Price of a call as a function of the spot (strike 100, volatility 30%, maturity 1 year, $r = 0$) and its tangent at $S = 100$. The slope of the tangent is the delta and its intercept $C - \delta S$ the cash part of the hedging portfolio.

Convexity. For a variation ΔS of the spot, a second order expansion gives

$$\Delta C \approx \delta \Delta S + \frac{1}{2} \Gamma (\Delta S)^2.$$

The term $\frac{1}{2} \Gamma (\Delta S)^2$ is the distance between the price curve and its tangent. Since $\Gamma > 0$ for calls and puts, an option always gains with respect to its hedging portfolio, whatever the direction of the move of the spot.

The hedged position, a little later. This gain is not free: as time passes, the option loses part of its time value. Figure 4 represents the P&L of a long call hedged with δ stocks, one month later, as a function of the new spot. The position loses locally, for small moves of the spot, because of the time decay, and gains on the wings, because of the convexity. With $r = 0$, the Black-Scholes equation reads $\Theta = -\frac{1}{2} \sigma^2 S^2 \Gamma$, so that, over a period dt ,

$$\text{P\&L} \approx \frac{1}{2} \Gamma \left((\Delta S)^2 - \sigma^2 S^2 dt \right) :$$

the break-even points are approximately at $S \left(1 \pm \sigma \sqrt{dt} \right)$. The hedged option gains if the spot moves more than what the volatility σ used for pricing anticipates, and loses otherwise.

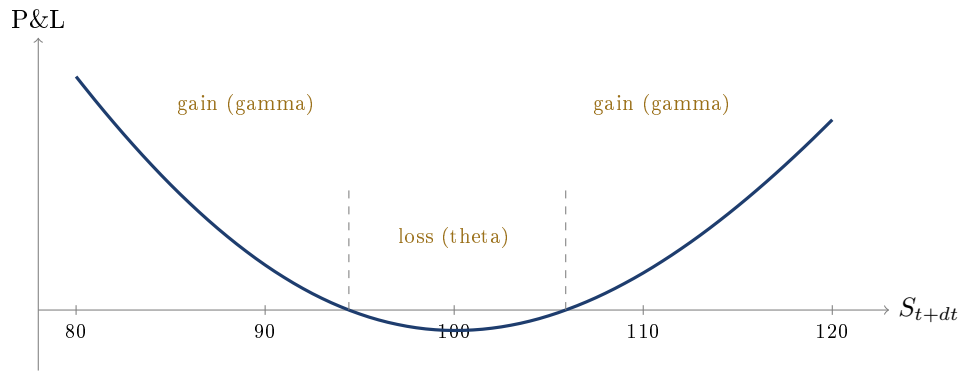


Figure 4: P&L, one month later, of a long call hedged at $S = 100$ with δ stocks (strike 100, volatility 20%, maturity 1 year, $r = 0$), as a function of the new spot. The break-even points are close to $100 \left(1 \pm 20\% \sqrt{1/12}\right)$.

8.2 Synthetic summary

Parameter	Greek	B&S (Call)	Call Option			Put Option		
			IV	TV	Price	IV	TV	Price
Delta	$\delta = \frac{\partial P}{\partial S}$	$N(d_1)$	$\nearrow$	$\nearrow \searrow$	$\nearrow$	$\searrow$	$\nearrow \searrow$	$\searrow$
Gamma	$\gamma = \frac{\partial^2 P}{\partial S^2}$	$\frac{n(d_1)}{S\sigma\sqrt{\tau}}$	$-\uparrow -$	$\nearrow \searrow$	$\nearrow \searrow$	$-\uparrow -$	$\nearrow \searrow$	$\nearrow \searrow$
Vega	$\mathcal{V} = \frac{\partial P}{\partial \sigma}$	$S n(d_1) \sqrt{\tau}$	$-$	$\nearrow$	$\nearrow$	$-$	$\nearrow$	$\nearrow$
Theta	$\Theta = \frac{\partial P}{\partial t}$	$-\frac{S n(d_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}N(d_2)$	$\searrow$	$\searrow *$	$\searrow$	$\nearrow$	$\searrow *$	$\nearrow \searrow$
Rho	$\rho = \frac{\partial P}{\partial r}$	$K\tau e^{-r\tau}N(d_2)$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
-	$\frac{\partial P}{\partial K}$	$-e^{-r\tau}N(d_2)$	$\searrow$	$?$	$\searrow$	$?$	$\searrow$	$\searrow$

Table 4: Sensitivities of call and put prices to the parameters of the Black-Scholes model. Arrows indicate the direction of the change when the parameter increases (IV = Intrinsic Value, TV = Time Value, $\tau = T - t$ is the time to maturity, n is the standard Gaussian density).

* Almost always true (do the calculations).

8.3 Different approaches

The greeks can be approached in three complementary ways:

- **Graphical representation:** as in Figures 3 and 4, the delta is the slope of the price curve, the gamma its curvature, and the theta the downward shift of the curve as time passes.
- **PDE approach:** the Black-Scholes equation (16) reads, in terms of greeks,

$$\Theta + rS\delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC.$$

With $r = 0$, $\Theta = -\frac{1}{2}\sigma^2 S^2 \Gamma$: time decay and convexity are two sides of the same coin.

- **Rederivation of the B&S formula:** computing $e^{-rT}E^Q\left((S_T - K)^+\right)$ with a log-normal S_T gives the B&S formula, and differentiating it gives the closed-form greeks of Table 4. For instance, using the identity $S n(d_1) = Ke^{-r\tau}n(d_2)$, one finds $\delta = N(d_1)$.

8.4 Exercises

Exercise 8.1

Joe believes that the volatility of a stock is higher than indicated by the market prices of options on that stock. He wants to speculate on that belief by buying or selling at-the-money options. Determine which of the following strategies would achieve Joe's goal (see forwardoptionpayoffs.pdf):

1. buy a strangle;
2. buy a straddle;
3. sell a straddle;
4. buy a butterfly spread;
5. sell a butterfly spread.

9 Options Use Cases

9.1 Building different payoffs

See: https://en.wikipedia.org/wiki/Black%E2%80%93Scholes_model.

Call Spread, Digital Option, straddle options

9.2 Principal Protected Products

ZC + Call, Spot + Put

9.3 Return enhancement

Selling calls, Selling puts

9.4 Exercises

Exercise 9.1 *Problem: a principal protected product*

We want to create a portfolio whose final value would be at least equal to its initial value V_0 . To do so, we aim at buying a zero-coupon bond and some call options on the S&P 500. Let r be the riskless interest rate, T the maturity, σ the volatility of the S&P 500 and d its dividend yield.

- (a) What notional amount of zero-coupon bond should be bought to ensure principal protection? Let D_0 be the initial value of this zero-coupon bond. Express D_0 as a function of V_0 , r and T . Define the “cushion” as $C_0 \equiv V_0 - D_0$. Compute C_0 when $r = 4\%$ and $T = 4$.
- (b) We consider buying call options on the S&P 500. Let S_0 be the price of the S&P today and K the selected strike price. We want to obtain the following payoff:

- if $S_T > S_0$: $kV_0 \left(\frac{S_T}{S_0} - 1 \right)$;
- if $S_T < S_0$: 0.

Write the payoff. How many calls, with which characteristics, do you need to obtain this payoff? Determine k such that the price of those calls equals C_0 .

- (c) Find k when $r = 4\%$, $T = 4$, $\sigma = 15\%$ and $d = 1.88\%$. Describe the payoff qualitatively. Compare with buying the S&P directly. Why is this not an arbitrage?

Appendix

Some examples of reliable data sources

Source	Comments	Web url	Free API /  package
Wikipedia	A wealth of information about virtually everything.	https://www.wikipedia.org	Semantic Web, dbpedia, wikidata, etc.  : Glitter
Bank of International Settlements	Plenty of data about trading volumes, bank capital etc.	https://www.bis.org	 BISdata
World Bank	World macro economic data	https://data.worldbank.org	 WDI
ISO	Standardized names for countries, currencies, exchanges etc.	http://www.iso.org	 ISOcodes
Yahoo finance	Stock prices historical High/Low/Close and Volume data	https://finance.yahoo.com/	 yahoofinancer
The Guardian	A reference UK based daily newspaper	https://www.theguardian.com/	https://open-platform.theguardian.com/

Table 5: Some reliable data sources

Solutions

Solutions to the Exercises

Solution 2.1

6.37%.

Solution 2.2

2.90%.

Solution 2.3

- (a) Since Air France will have to pay USD in one year, its risk is that the USD appreciates against the EUR.
- (b) Buying the USD forward costs, in euros, $V_1 = \$300,000,000/1.112 = \text{€}269,784,173$.
- (c)
- Lending USD: $300,000,000 \times e^{-0.05} = \$285,368,827$;
 - Borrowing the corresponding euros and changing them spot: $\$285,368,827/1.09 = \text{€}261,806,264$;
 - Repaying the euro loan in one year: $\text{€}261,806,264 \times e^{0.03} = \text{€}269,779,451$.

Both alternatives cost the same, up to the rounding of the forward rate: this is the cash and carry argument of the forward exchange rate.

Solution 2.4

Questions (a) to (c). The three strategies can be parametrised by t , the maturity of the first loan. At t , instead of repaying the loan, one borrows again until date T . If N is the notional amount, the amount to be paid at date t is $\frac{N}{P(0,t)}$ (which is known). At date t , this amount has to be borrowed again until date T , when $\frac{N}{P(0,t)} \frac{1}{P(t,T)}$ will have to be repaid. As a result, the NPV of these strategies is

$$NPV(t) = N \left(1 - \frac{P(0,T)}{P(0,t)P(t,T)} \right).$$

Note that $P(t,T)$ is random since $r(t,T)$ is not known today. In terms of interest rates,

$$NPV(t) = N \left(1 - e^{-r(0,T)T + r(0,t)t + r(t,T)(T-t)} \right).$$

The NPV is positive if and only if $-r(0,T)T + r(0,t)t + r(t,T)(T-t) < 0$, i.e.

$$r(t,T) \leq f(t,T) \quad \text{where} \quad f(t,T) := \frac{r(0,T)T - r(0,t)t}{T-t}.$$

The NPV is positive only if the future rate at date t is below the forward rate.

Question (d). If the yield curve does not move, $r(t,T) = r(0,T-t)$. Remark that

$$f(t,T) = r(0,T) + t \frac{r(0,T) - r(0,t)}{T-t}.$$

If the yield curve is upward sloping,

$$\begin{aligned} r(0,T) &\geq r(0,T-t) \quad (\text{because } T > T-t) \\ r(0,T) - r(0,t) &\geq 0 \end{aligned}$$

hence $f(t,T) \geq r(0,T-t)$. So, when the yield curve does not move, the realised interest rate is below the forward rate and the strategy is profitable.

Question (e).

	$T-t=1$	$T-t=2$	$T-t=3$	$T-t=4$
$t=0$	3%	3.3%	3.6%	3.7%
$t=1$	3.6%	3.9%	3.93%	NA
$t=2$	4.2%	4.1%	NA	NA
$t=3$	4.0%	NA	NA	NA

Questions (f) and (g). To obtain N euros today, you need $\frac{N}{Z(0)}$ yen, which you can borrow until date T . You will have to repay $\frac{N}{Z(0)} \frac{1}{P^{yen}(0,T)}$ yen, which costs $N \frac{Z(T)}{Z(0)} \frac{1}{P^{yen}(0,T)}$ euros. As a result, the NPV is

$$\begin{aligned} NPV &= N \left(1 - \frac{Z(T)}{Z(0)} \frac{P^{eur}(0,T)}{P^{yen}(0,T)} \right) \\ &= N \left(1 - \frac{Z(T)}{F(0,T)} \right) \end{aligned}$$

where $F(0,T) := Z(0) \frac{P^{yen}(0,T)}{P^{eur}(0,T)} = Z(0) e^{T(r^{eur}(T) - r^{yen}(T))}$ is the forward exchange rate. The NPV is positive if and only if

$$Z(T) < F(0,T),$$

i.e. if the future exchange rate is lower than the forward exchange rate.

In our example, $P^{yen}(0,T) = 1$. If the yen follows the trend indicated by the graph, $Z(4)$ might not increase, and could very well decrease, generating a gain. If the yen depreciates by 30% over the coming 4 years, $Z(4) = 70\%Z(0)$ and the NPV is above 30% of the notional.

Solution 3.1

- (a) 83.06 USD
- (b) 80.02 USD
- (c) 79.73 USD

Solution 3.2

- (a) 55.17 USD
- (b) 57.12 USD
- (c) 59.97 USD
- (d) 59.97 USD

Solution 3.3

- (a) Let D be the value of the dividend. By absence of arbitrage, $1109 = \frac{1100}{0.97} - D$, hence $D = 1134 - 1109 = 25$.
- (b) If the dividend paid is 15 instead of 25, the forward price should be $1134 - 15 = 1119$: the forward is too cheap. One should go long the forward, sell the index short and lend the proceeds, which yields a profit of 10 at maturity.

Solution 4.1

With zero interest rates, borrowing costs nothing and the futures price of the Nikkei is equal to its spot price. The Nikkei multiplies by 1.57 in JPY and the USD/JPY moves from 87 to 105.

- (a) Strategy 1: the USD 1M becomes JPY 87M, invested in the Nikkei and worth JPY $87 \times 1.57 = 136.59$ M at the end of the year, i.e. USD $136.59/105 = 1.301$ M. After repaying the USD loan, the P&L is +USD 0.301M (+30%): the rise of the Nikkei is largely eaten up by the fall of the yen, since the whole position is exposed to the currency.

Strategy 2: the JPY 87M loan is invested in the Nikkei, worth JPY 136.59M at the end of the year. After repaying the JPY loan, JPY 49.59M remain, i.e. USD $49.59/105 = 0.472$ M (+47%). Borrowing in yen hedges the principal against the currency risk: only the gain is exposed to the fall of the yen.

- (b) Strategy 3: the long futures position on the Nikkei pays $87 \times 0.57 =$ JPY 49.59M at expiry, i.e. USD 0.472M: it is equivalent to strategy 2 (a futures contract is a position financed in the currency of the underlying).

Strategy 4: the long position on USD/JPY pays $1 \times (105 - 87) =$ JPY 18M, i.e. USD $18/105 = 0.171$ M (+17%).

The two strategies can be combined, since neither requires an initial investment: the P&L is then USD $0.472 + 0.171 = 0.644$ M.

- (c) The appropriate “trade” was to **buy the Nikkei and sell the yen**, both through futures: long Nikkei futures and long USD/JPY futures (i.e. short yen). The equity view and the currency view are then implemented separately, without being exposed to the fall of the yen on the equity position: each leg earns its own performance.

Solution 6.1

The appropriate formula is the Black formula for futures. It can be obtained from a Black and Scholes calculator with the following parameters:

- spot = forward = 25;
- strike, maturity and rate at their level;
- dividend yield = rate, to ensure that the forward price is equal to the spot price.

The price is 2.53.

Solution 6.2

- (a) 4.832
- (b) 4.176

Solution 6.3

- (a) The forward price is a quantity known today (t), paid at date T in exchange for a random quantity, with no initial exchange of money.
- (b) Cash and carry arbitrage, see the section on the forward exchange rate.
- (c) Apply the Black formula with $t = 0$, $r = r_{\text{€}}$ and the forward price:

$$C = e^{-r_{\text{€}}T} [F_T N(d_1) - K N(d_2)]$$

where

$$\begin{aligned} F_T &= X_0 e^{(r_{\text{€}} - r_{\text{\$}})T} \\ d_1 &= \frac{\ln(F_T/K) + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} \\ d_2 &= d_1 - \sigma \sqrt{T} \end{aligned}$$

Hence, the foreign rate plays the role of a dividend yield.

- (d) The value of the call is always above its intrinsic value (convexity):

$$C > (X_0 e^{-r_{\text{\$}}T} - K e^{-r_{\text{€}}T})^+.$$

The value obtained when exercising an American call is

$$C_{ex} = X_0 - K.$$

For options on stocks without dividends, necessarily $C > C_{ex}$, so that it is always better to sell the call than to exercise it. In the FX case, if the foreign rate $r_{\text{\$}}$ is high enough, the intrinsic value can fall below $X_0 - K$, and it might be better to exercise the call than to sell it.

Solution 6.4

Because interest rates are null, the price of the zero-coupon bond with notional K is K . By the call-put parity,

$$67 - 19 = 500 - K$$

hence $K = 452$.

Solution 6.5

A long call and a short put with the same strike provide an exposure to one unit of the index, together with a borrowing of the present value of the strike. By the call-put parity, the cost is

$$Call - Put = 1000 - \frac{1025}{1.05} \simeq 23.81,$$

which corresponds to answer 4. The strike which allows to secure the purchase at zero cost is the forward price:

$$1000 - \frac{K}{1.05} = 0 \Leftrightarrow K = 1050.$$

Solution 6.6

Since we are close to the maturity, we can make the approximation $Ke^{-rT} \simeq K$. We can thus complete the table and compute which stock price would allow the call-put parity to hold:

Strike price	Call price	Put price	Call - Put + Strike
40	11	3	48
50	6	8	48
55	3	11	47

Since the stock price cannot be equal to both 48 and 47, this option price table is inconsistent and can be arbitrated. An arbitrage strategy consists of buying what is cheap and selling what is expensive. By the call-put parity, a (long call, short put, long zero-coupon) position replicates the underlying.

With strike 55, this costs 47, whereas with strike 50 (or 40) it costs 48. Buying this position with strike 55 and selling it with strike 40, you receive 1 today and have no other cash flow: this is an arbitrage.

Solution 6.7

The cash and carry arbitrage consists of borrowing in local currency, changing spot, carrying in foreign currency (hence receiving the foreign short rate) and changing back at maturity. As a result:

$$F(t, T) = X_t e^{(r_l - r_f)(T-t)}.$$

The forward price is the expectation of the price at T under the risk neutral probability: $E(X_T | X_t) = F(t, T)$. If X_t is modelled as $dX_t = X_t(\mu dt + \sigma dW_t)$, then

$$X_T = X_t e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma(W_T - W_t)}$$

and hence

$$E(X_T | X_t) = X_t e^{\mu(T-t)}.$$

So, under the risk neutral probability, necessarily $\mu = r_l - r_f$.

Solution 6.8

Let $\tilde{S}_T$ be the (random) price of a given stock at date T , S_0 its price today, and K a given amount of money.

Question (a). For any value of $\tilde{S}_T$,

$$\tilde{S}_T + (K - \tilde{S}_T)^+ = K + (\tilde{S}_T - K)^+ \quad (22)$$

since

- when $\tilde{S}_T \leq K$, both sides are equal to K ;
- when $\tilde{S}_T \geq K$, both sides are equal to $\tilde{S}_T$.

If there are no intermediate flows such as coupons or dividends, by absence of arbitrage (sometimes referred to as “the law of one price”), the prices of both sides of equation (22) are equal. Since the price is linear:

$$\underbrace{\text{Price}(\tilde{S}_T)}_1 + \underbrace{\text{Price}\left((K - \tilde{S}_T)^+\right)}_2 = \underbrace{\text{Price}(K)}_3 + \underbrace{\text{Price}\left((\tilde{S}_T - K)^+\right)}_4$$

This proves the statement since:

1. is the price of the underlying, S_0 ;
2. is the price of the put, $Put(S_0, K, T, r)$;
3. is the price of a zero-coupon bond with notional K ;
4. is the price of the call, $Call(S_0, K, T, r)$.

Question (b). In the Black and Scholes model,

$$dS_t = S_t(r dt + \sigma dW_t)$$

under the risk neutral probability. By definition, the forward price is $F_t = S_t e^{r(T-t)}$, and Itô's lemma yields:

$$\begin{aligned} dF_t &= e^{r(T-t)} dS_t + S_t d(e^{r(T-t)}) \\ &= F_t(r dt + \sigma dW_t) - F_t r dt \\ &= F_t \sigma dW_t \end{aligned}$$

which shows that the forward price has a null drift (it is a martingale) under the risk neutral probability.

Question (c). This question is very similar to the first one, but instead of taking the price of both sides of equation (22), we take their risk neutral expectation:

$$\underbrace{\mathbb{E}(\tilde{S}_T)}_1 + \underbrace{\mathbb{E}\left((K - \tilde{S}_T)^+\right)}_2 = \underbrace{\mathbb{E}(K)}_3 + \underbrace{\mathbb{E}\left((\tilde{S}_T - K)^+\right)}_4$$

Term 1 is the expected value of the spot at date T , i.e. its forward value $F = S_0 e^{rT}$. Term 2 can be related to a B&S formula with modified parameters:

$$\begin{aligned}\mathbb{E} \left(\left(K - \tilde{S}_T \right)^+ \right) &= e^{rT} \text{Put} (S_0, K, T, r) \\ &= KN(-d_2) - FN(-d_1)\end{aligned}\quad (23)$$

with

$$\begin{aligned}d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ &= \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}.\end{aligned}\quad (24)$$

Expressions (23) and (24) show that $\mathbb{E} \left(\left(K - \tilde{S}_T \right)^+ \right)$ is the value of a put option on a futures contract in a world with no interest rates, i.e. $\text{Put}(F, K, T, 0)$. The same reasoning applies to the call. Finally, $\mathbb{E}(K) = K$, which proves the statement.

Solution 6.9

Question (a). Shareholders are long a call option on the firm value at date T , with strike the amount of debt to be repaid:

$$(S_T - D)^+.$$

Hence, using the B&S formula, the value of the equity is

$$C(T, F_T) = e^{-rT} F_T [N(d_1) - lN(d_2)]$$

where

$$\begin{aligned}F_T &= S_0 e^{rT} \\ d_1 &= \frac{-\ln(l) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \\ d_2 &= \frac{-\ln(l) - \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}\end{aligned}$$

Question (b). Since assets are equal to liabilities, the final payoff of creditors is

$$D^r = S_T - (S_T - D)^+.$$

By the call-put parity, its value is

$$D^r = D e^{-rT} - P(T, F_T)$$

where

$$P(T, F_T) = e^{-rT} F_T [lN(-d_2) - N(-d_1)].$$

Hence

$$\begin{aligned}D e^{-(r+s)T} &= e^{-rT} F_T [l - lN(-d_2) + N(-d_1)] \\ \iff e^{-sT} &= N(d_2) + \frac{1}{l} N(-d_1)\end{aligned}$$

$$\boxed{s = -\frac{1}{T} \ln \left(N(d_2) + \frac{1}{l} N(-d_1) \right)}$$

When leverage goes to zero:

$$\begin{aligned}l \rightarrow 0 : -\ln(l) &\rightarrow +\infty, N(d_2) \rightarrow 1, \frac{1}{l} N(-d_1) \rightarrow 0 \\ \Rightarrow s &\rightarrow 0\end{aligned}$$

When leverage goes to 1, the debt is equal to the forward value of the assets:

$$l \rightarrow 1 : d_1 \rightarrow \frac{\sigma\sqrt{T}}{2}, d_2 \rightarrow -\frac{\sigma\sqrt{T}}{2}$$

$$\Rightarrow s \rightarrow -\frac{1}{T} \ln \left(2N \left(-\frac{\sigma\sqrt{T}}{2} \right) \right)$$

which is finite (about 400bp for $\sigma = 10\%$ and $T = 1$) but increases with the volatility.

Question (c). With $r = 0$, $F_T = S_0$ and values are expressed as a fraction of S_0 :

σ	l	equity	risky debt	spread s
20%	1/2	0.500	0.500	≈ 0 bp
20%	3/4	0.256	0.744	78bp
40%	1/2	0.505	0.495	95bp

With low leverage and low volatility, the probability of bankruptcy is negligible: equity is worth $1 - l$ and the debt is riskless. The spread increases both with leverage and with volatility.

Question (d). When σ increases, the value of the call increases, hence the value of equity increases. The value of the put increases as well, hence the value of the risky debt decreases: shareholders have an interest in the company taking more risk, at the expense of creditors.

Question (e). Before the dividend, the equity is worth $0.273 S_0$ and the spread is 315bp. The dividend reduces the value of the company to $0.95 S_0$, hence its forward value to $F'_T = 0.95 F_T$ and the leverage increases to $l' = \frac{3/4}{0.95} \simeq 0.79$. The equity is then worth $0.231 S_0$ and the spread increases to 421bp. Shareholders receive $0.05 S_0 + 0.231 S_0 = 0.281 S_0$, more than the $0.273 S_0$ they had: the dividend transfers wealth from creditors to shareholders. The risky debt is now worth $0.719 S_0$ instead of $0.727 S_0$.

Solution 6.10

- (a) If the USD appreciates, one gets more euros for 1 USD, hence X increases. If the CEO is right, X should increase and MNC should not hedge. If the CFO is right, the opposite is true.
- (b) Since USD rates are higher than euro rates, the forward rate is lower than the spot rate:

$$F_{3m} = X_0 e^{(r_{EUR} - r_{USD})T}$$

$$= 0.9 \times e^{-4\%/4}$$

$$\approx 0.9 \times (1 - 1\%)$$

$$\approx 0.891$$

In EUR, this means about EUR 89.1 million instead of EUR 90 million, i.e. a difference of approximately EUR 0.9 million. If the future value of the spot equals the forward, the USD will have depreciated a bit.

- (c) The variance of a process with i.i.d. increments is of the form $\sigma^2 T$, hence the standard deviation (volatility) scales as $\sigma\sqrt{T}$. The quarterly standard deviation is thus half of the yearly volatility, i.e. 5%. The typical fluctuation in 3 months is therefore $\pm 5\%$ around EUR 89.1 million, i.e. between approximately EUR 84.6 million and EUR 93.6 million.
- (d) No risk means a fixed amount of EUR known today: sell the USD 100 million forward immediately. The CFO would approve, but the CEO would think that MNC is losing the opportunity to benefit from a potential appreciation of the USD against the EUR.
- (e) The risk of the company is a decrease of X , hence a put option is suitable.
- (f) See Figure 5.

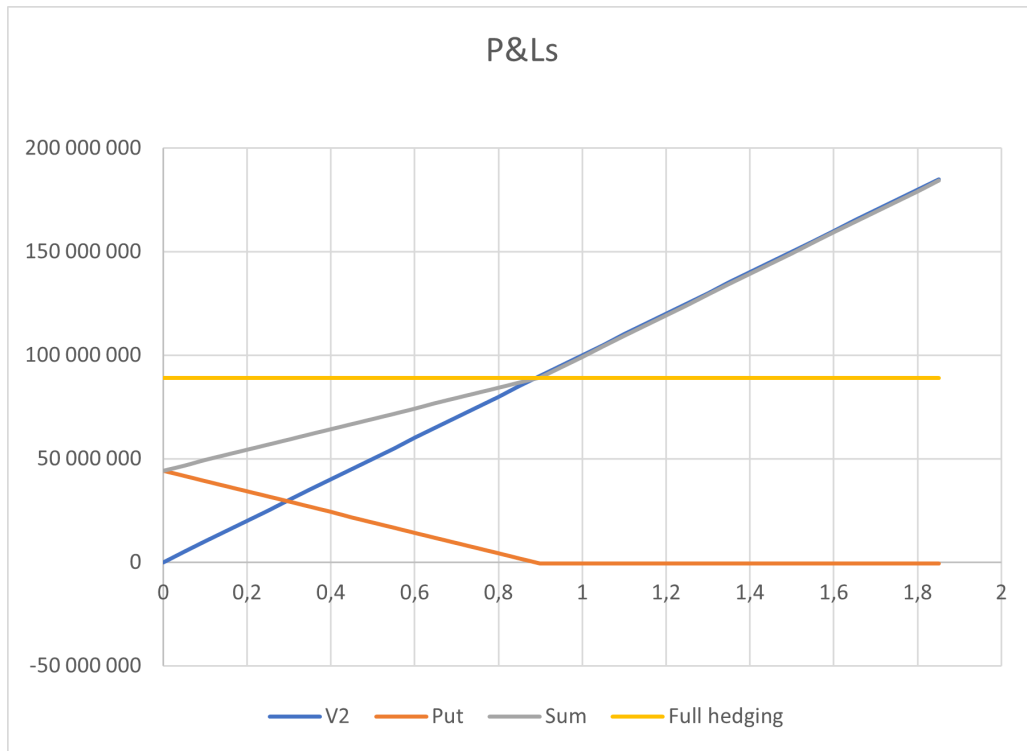


Figure 5: Solution of the FX problem

- (g) With a full hedge, we would obtain the yellow curve. In my opinion, we do not have specific prediction capabilities, and the risk should be hedged.

Solution 8.1

Joe has to be long volatility, i.e. long options. The best strategy is 2 (buy an at-the-money straddle, which has the highest exposure to volatility). The second best is 1 (a strangle, with out-of-the-money options, is less exposed). Selling a straddle (3) or buying a butterfly spread (4) are short volatility positions. Selling a butterfly spread (5) is long volatility, but its gains are capped.

Solution 9.1

- (a) $D_0 = V_0 e^{-rT}$, hence

$$C_0 = V_0 (1 - e^{-rT}).$$

In the example, with $V_0 = 1$, $C_0 = 14.79\%$.

- (b) If one buys n at-the-money calls, the payoff is

$$n(S_T - S_0)^+$$

so that with $n = \frac{kV_0}{S_0}$ we obtain

$$kV_0 \left(\frac{S_T}{S_0} - 1 \right)^+$$

which is the required payoff. k is then obtained by the budget equality:

$$C_0 = kV_0 \text{Call}(S_0 = 1, K = 1, \sigma, T)$$

$$k = \frac{1 - e^{-rT}}{\text{Call}(S_0 = 1, K = 1, \sigma, T)}$$

- (c) The price of the call equals 14.8%, hence $k = 100\%$. This product provides the performance of the S&P 500 when it is positive, and zero (i.e. the capital back) when it is negative. It looks like an arbitrage but it is not: the structured product does not receive the dividends, whereas a direct investment in the S&P 500 would.

References

- [1] John C Cox, Jonathan E Ingersoll Jr, and Stephen A Ross. The relation between forward prices and futures prices. *Journal of Financial Economics*, 9(4):321–346, 1981.
- [2] John Hull. *Options, futures, & other derivatives. Solutions manual*. Prentice Hall International, 2006.
- [3] Robert C Merton. Theory of rational option pricing. *The Bell Journal of economics and management science*, pages 141–183, 1973.
- [4] Myron Scholes and Fischer Black. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.